AT721 Section 9:

Characteristic solutions: II Anisotropic scatter

General reference: Benedetti et al., 2002;Properties of reflected sunlight derived from Green's function method, JQRST, 201-225

9.1 The canonical form of the homogeneous solution

We start with the matrix of the radiative transfer equation

$$\frac{d}{d\tau}\begin{pmatrix} I^+ \\ I^- \end{pmatrix} = \begin{pmatrix} -t & r \\ -r & t \end{pmatrix}\begin{pmatrix} I^+ \\ I^- \end{pmatrix} \tag{9.1a}$$

or as

$$\frac{d\mathbf{I}}{d\tau} = \mathbf{A}\ \mathbf{I} \tag{9.1b}$$

where the sub–matrices r and t have been defined earlier for the discrete (i.e. quadrature) μ space. The solution to (9.1b) is trivial when the matrix $\mathbf{A}$ is reduced to a diagonal form Λ. That is, suppose we can find a $2n \times 2n$ invertible matrix $\mathbf{X}$ such that

$$\mathbf{A} = \mathbf{X}\ \Lambda\ \mathbf{X}^{-1} \tag{9.2}$$

where

$$\Lambda = \begin{pmatrix} k_1 & 0 & \dots \\ 0 & \ddots & 0 \\ 0 & \dots & k_{2n} \end{pmatrix}$$

and where both $\mathbf{X}$ and k_j define the eigen–structures of $\mathbf{A}$, namely

$$\mathbf{A}\ \mathbf{X} = \mathbf{X}\ \Lambda \tag{9.3}$$

which requires that $\mathbf{X}$ be composed of the eigenvectors in the following way

$$\mathbf{X} = [\underline{u}_1, \dots, \underline{u}_n, \underline{u}_{n+1}, \dots, \underline{u}_{2n}] \tag{9.4}$$

where $\underline{u}_j$ is the j^{th} (column) eigenvector associated with the k_j^{th} eigenvalue. If we write (9.1b) as

$$\mathbf{X}^{-1}\frac{d\mathbf{I}}{d\tau} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}\mathbf{X}^{-1}\mathbf{I}$$

then it follows from (9.3) that

$$\mathbf{X}^{-1}\frac{d\mathbf{I}}{d\tau} = \Lambda\mathbf{X}^{-1}\mathbf{I}$$

and by introducing $\hat{\mathbf{I}} = \mathbf{X}^{-1}\mathbf{I}$, then

$$\frac{d\hat{\mathbf{I}}}{d\tau} = \Lambda\hat{\mathbf{I}} \tag{9.5}$$

with the solution

$$\hat{\mathbf{I}}(\tau) = \exp(\Lambda\tau)\hat{\mathbf{I}}(0) \tag{9.6}$$

where $\exp(\Lambda\tau)$ is a diagonal matrix and its evaluation is straight forward, i.e.

$$\exp(\Lambda\tau) = \begin{pmatrix} \exp(k_1\tau) & 0 & \dots \\ 0 & \ddots & 0 \\ 0 & \dots & \exp(k_{2n}\tau) \end{pmatrix}$$

Multiplying (9.6) by the matrix $\mathbf{X}$ transforms the quantity $\hat{\mathbf{I}}$ to a solution in terms of physical radiance I, namely

$$\mathbf{I}(\tau) = \mathbf{X}\exp(\Lambda\tau)\mathbf{X}^{-1}\mathbf{I}(0) \tag{9.7a}$$

from which we identify

$$\mathbf{M}(0,\tau) = \exp(\mathbf{A}\tau) = \mathbf{X}\exp(\Lambda\tau)\mathbf{X}^{-1} \tag{9.7b}$$

The matrix $\mathbf{M}(0,\tau)$ is referred to as a *mapping* property or propagator as it maps the $2n$ radiance vector $\mathbf{I}(0) \rightarrow \mathbf{I}(\tau)$. Thus the solution of the general radiative transfer problem reduces to the evaluation of this matrix which can be achieved through computation of the eigenvalues and associated eigenvectors of $\mathbf{A}$ according to (9.7). This task poses certain numerical difficulties that can be relieved in part by exploiting certain properties of the structure of the $\mathbf{A}$ matrix discussed below.

9.2 The homogeneous solution

Owing to the form of $\mathbf{A}$ as given by (4.22) then the matrix $\mathbf{X}$ of eigenvectors has the following block structure

$$\mathbf{X} = \begin{pmatrix} \mathbf{u}_+ & \mathbf{u}_- \\ -\mathbf{u}_- & -\mathbf{u}_+ \end{pmatrix} \tag{9.8}$$

where $\mathbf{u}_\pm$ are $n \times n$ block matrices constructed from the eigenvectors as mentioned above in the following way

$$\mathbf{u}_\pm = [\underline{u}_1^\pm.....\underline{u}_n^\pm]$$

It can be shown that the inversion of $\mathbf{X}$ also partitions in a manner similar to (9.8), namely

$$\mathbf{X}^{-1} = \begin{pmatrix} \mathbf{v}_+ & \mathbf{v}_- \\ -\mathbf{v}_- & -\mathbf{v}_+ \end{pmatrix} \tag{9.9}$$

such that the block matrices $\mathbf{v}_+$ and $\mathbf{v}_-$ follow as

$$\mathbf{u}_+\mathbf{v}_+ + \mathbf{u}_-\mathbf{v}_- = E_n$$

$$\mathbf{u}_+\mathbf{v}_- + \mathbf{u}_-\mathbf{v}_+ = 0_n$$

where E_n is the $n \times n$ identity matrix and 0_n is the $n \times n$ zero matrix. In making use of the block nature of the $\mathbf{A}$ matrix , these eigenmatrices can be efficiently determined in the manner below. It thus follows that

$$\mathbf{v}_+ = [E_n - (\mathbf{u}_+^{-1}\mathbf{u}_-)^2]^{-1}\mathbf{u}_+^{-1}$$

$$\mathbf{v}_- = [\mathbf{u}_+^{-1}\mathbf{u}_+]\mathbf{v}_+ \tag{9.10}$$

Introducing (9.10), (9.11) in (9.7), we obtain

$$\mathbf{M} = \begin{pmatrix} \mathbf{u}_+ & \mathbf{u}_- \\ -\mathbf{u}_- & -\mathbf{u}_+ \end{pmatrix} \begin{pmatrix} e^{\Lambda_+\tau} & 0 \\ 0 & e^{-\Lambda_+\tau} \end{pmatrix} \begin{pmatrix} \mathbf{v}_+ & \mathbf{v}_- \\ -\mathbf{v}_- & -\mathbf{v}_+ \end{pmatrix} \tag{9.11}$$

where Λ_+ is a $n \times n$ diagonal matrix of the positive eigenvalues of $\mathbf{A}$. It is clear that this mapping matrix also has a block structure, namely

$$\mathbf{M} = \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix} \tag{9.12}$$

such that

$$\begin{aligned} M_{++} &= \mathbf{u}_+ e^{\Lambda_+\tau}\mathbf{v}_+ - \mathbf{u}_- e^{-\Lambda_+\tau}\mathbf{v}_- \\ M_{-+} &= -\mathbf{u}_- e^{\Lambda_+\tau}\mathbf{v}_+ + \mathbf{u}_+ e^{-\Lambda_+\tau}\mathbf{v}_- \\ M_{+-} &= \mathbf{u}_+ e^{\Lambda_+\tau}\mathbf{v}_- - \mathbf{u}_- e^{-\Lambda_+\tau}\mathbf{v}_+ \\ M_{--} &= -\mathbf{u}_- e^{\Lambda_+\tau}\mathbf{v}_- + \mathbf{u}_+ e^{-\Lambda_+\tau}\mathbf{v}_+ \end{aligned} \tag{9.13}$$

follows from the expansion of (9.11). Given (9.6), (9.7) and (9.12) the general solution,

$$\mathbf{I}(\tau) = \exp(\mathbf{A}\tau)\mathbf{I}(0)$$

becomes

$$\begin{aligned} I^+(\tau) &= M_{++}I^+(0) + M_{+-}I^-(0) \\ I^-(\tau) &= M_{-+}I^+(0) + M_{--}I^-(0) \end{aligned} \tag{9.14}$$

9.2.1 The homogeneous solution in expanded form

Substituting (9.13) into (9.14) yields

$$\begin{aligned} I^+(\tau) = {} & [\mathbf{u}_+ e^{\Lambda_+\tau}\mathbf{v}_+ - \mathbf{u}_- e^{-\Lambda_+\tau}\mathbf{v}_-]I^+(0) \\ & + [\mathbf{u}_+ e^{\Lambda_+\tau}\mathbf{v}_- - \mathbf{u}_- e^{-\Lambda_+\tau}\mathbf{v}_+]I^-(0) \\ I^-(\tau) = {} & [-\mathbf{u}_- e^{\Lambda_+\tau}\mathbf{v}_+ + \mathbf{u}_+ e^{-\Lambda_+\tau}\mathbf{v}_-]I^+(0) \\ & + [-\mathbf{u}_- e^{\Lambda_+\tau}\mathbf{v}_- + \mathbf{u}_+ e^{-\Lambda_+\tau}\mathbf{v}_+]I^-(0) \end{aligned}$$

and with some rearrangement

$$\begin{aligned} I^+(\tau) &= \mathbf{u}_+ e^{\Lambda_+\tau}\mathbf{L}^+ - \mathbf{u}_- e^{-\Lambda_+\tau}\mathbf{L}^- \\ I^-(\tau) &= -\mathbf{u}_- e^{\Lambda_+\tau}\mathbf{L}^+ + \mathbf{u}_+ e^{-\Lambda_+\tau}\mathbf{L}^- \end{aligned} \tag{9.15}$$

where

$$\mathbf{L}^+ = [\mathbf{v}_+ I^+(0) + \mathbf{v}_I^-(0)]$$

$$\mathbf{L}^- = [\mathbf{v}_- I^+(0) + \mathbf{v}_+ I^-(0)]$$

In expanded form, (9.15) can be written

$$I^{\pm}(\tau,\mu_i) = \sum_{j=1}^{n} L_j^{+} e^{-k_j\tau} u_{ij}^{\pm} + \sum_{j=1}^{n} L_j^{-} e^{-k_j\tau} u_{ij}^{\pm} \tag{9.16}$$

TBD *********** Now consider a semi-infinite atmosphere, then it follows by assertion that $L_j^+ = 0$ for $j = 1, \cdots n$ and

We now consider what information about the radiation field in contained in the eigenvectors $e(\pm)$. Suppose we are deep in a medium somewhere say between depths 100 and 250m as in Fig. 6.5. The intensity field at these deep levels associated with an incident intensity along a given μ_j direction follows from (6.27) as

$$I^{\pm}(\tau \to \infty, \mu_i) = L_j^{-} e^{-k_j\tau} e_{ij}^{\pm}.$$

Therefore e_{ij} represents the scattered angular radiance field in the direction μ_i for light incident along direction μ_j deep in a medium. The **shape** of this distribution is invariant with depth and the magnitude involves the factors $L_j^{-} e^{-k_j\tau}$

TBD******

Again we note that the block structure of **A** gives rise to eigenvalues in signed pairs and we con reduce the $2n \times 2n$ problem to an $n \times n$ problem. To illustrate this, consider our $2n \times 2n$ eigenmatrix equation

$$\begin{pmatrix} -t & r \\ -r & t \end{pmatrix} \begin{pmatrix} u_j^+ \\ u_j^- \end{pmatrix} = k_j \begin{pmatrix} u_j^+ \\ u_j^- \end{pmatrix}$$

which can be rearranged (by expanding and adding and subtracting) to give the following characteristic equation

$$(t-r)(t+r)(u_j^- - u_j^+) = k_j^2 (u_j^- - u_j^+)$$

which can be solved for the n eigenvalues, k_j^2 (and thus the n $\pm$ pairs) and the associated eigenvectors $u_j^- - u_j^+$ where $j = 1, \ldots, n$. With these solutions we obtain $u_j^+ + u_j^-$ from the related equation

$$-(t-r)(u_j^+ + u_j^-) = k_j (u_j^+ - u_j^-)$$

and then finally u_j^+ and u_j^- from two pairs $u_j^- - u_j^+$ and $u_j^+ + u_j^-$.

9.3 The interaction principle

We begin by rearranging (9.15) into its usual interaction form, namely

$$\begin{pmatrix} I^+(0) \\ I^-(\tau) \end{pmatrix} = \begin{pmatrix} -M_{++}^{-1} M_{+-} & M_{++}^{-1} \\ -M_{-+} M_{++}^{-1} M_{+-} + M_{--} & M_{-+} M_{++}^{-1} \end{pmatrix} \begin{pmatrix} I^-(0) \\ I^+(\tau) \end{pmatrix}$$

from which we can readily identify the global reflection and transmission functions as

$$\begin{aligned} R(0,\tau) &= -M_{++}^{-1}M_{+-} \\ T(\tau,0) &= M_{++}^{-1} \\ R(\tau,0) &= M_{-+}M_{++}^{-1} \\ T(0,\tau) &= M_{--} - M_{-+}M_{++}^{-1}M_{+-} \end{aligned} \qquad eqno(9.18)$$

9.3.1 The Stable Form

(ref Benedetti et al, 2002)

The form of the reflection and transmission matrices of (9.18) are unstable in conditions of large τ. As described in Benedetti et al, these matrix functions can we rearranged into a naturally stable form as $\tau \to \infty$. The form of these functions become

$$R(\tau,0) = -\mathbf{u}_+[\mathbf{E} - (\mathbf{u}_+^{-1}\mathbf{u}_-)e^{-\Lambda_+\tau}(\mathbf{u}_+^{-1}\mathbf{u}_-)^{-1}e^{-\Lambda_+\tau}]\times$$
$$[\mathbf{E} - [(\mathbf{u}_+^{-1}\mathbf{u}_-)^{-1}e^{-\Lambda_+\tau}]^2]^{-1}\mathbf{u}_-$$

etc

9.4 Properties of the homogenous solution

9.4.1 A Physical Perspective

TBD An attempt is now made to provide a physical setting to interpret this canonical form of solution. In particular, we will attempt to answer the question: what is the physical basis for the diagonal matrix Λ and for the vectors $\underline{u}_j$ comprising $\mathbf{X}$. Further, what physical reason may be given for the invertibility of $\mathbf{X}$.

The basis for interpreting the mathematical structure of (9.6a) and (9.6b) and it's solution couched in terms of eigenstructures may be introduced by considering a plot of the variation of the intensity as a function of depth into the medium as shown in Fig. 9.1. One sees from such a plot that the change in the logarithm of irradiance with depth becomes linear from some depth. If the medium is homogeneous and very deep and since our equation is linear, then this suggests that perhaps there may be some linear combination of observable vectors that decays or grows precisely as an exponential function of (optical) depth – and that these exponentially growing and decaying functions linearly combine to yield this observable intensity vector. This is merely further endorsement of the assumptions introduced in relation to the approximations described in chapter 7 and further a reflection of the expanded form of solution (e.g (9.16)) 8.x) in which the solution separates in τ and μ.

Furthermore, Fig. 9.1 also emphasizes that the τ factor is exponential in form. Given this observation, let us suppose the existence of $2n$ distinct exponential functions in optical depth y over the range $x \leq y \leq z$

$$B_j^{\pm}(y) \equiv B_j^{\pm}(x)\exp\left[k_j^{\pm}(y-x)\right] \qquad j = 1,\ldots,n \tag{9.19}$$

where $k_j^{\pm}, j = 1, ..., n$ are $2n$ distinct real numbers * which come as pairs k_j^+ for $j = 1,\ldots,n$, and k_j^- with $j = n+1,\ldots,2n$.

* Recall that y in the body of this study is optical depth, so that $y = \sigma_{ext}z$ where z is the associated geometric depth and σ_{ext} is the volume attenuation coefficient. Hence the $k_j^{\pm}$'s are dimensionless.

Figure 9.1 The depth dependence of the downward irradiance showing an almost linear dependence beyond 50m. The occurrence of discrete flashes generated by Euphasia pacifica at depths around 300–500m can also been seen (Preisendorfer, 1976).

Then we may write the intensity as a combination of $2n$ functions

$$I^{\pm}(y,\mu_i) = \sum_{j=1}^{n} B_j^{+}(y) f_j^{+^{\pm}}(\mu_i) + \sum_{j=1}^{n} B_j^{-}(y) f_j^{-^{\pm}}(\mu_i) \tag{9.21}$$

for $\mu_i : i = 1,\ldots,n$ and $x \le y \le z$ and where the $f_j^{+^{\pm,}}$'s are coefficients determined from boundary conditions.

9.4.2 The eigenvectors, asymptotic radiances and the diffusion pattern

TBD We consider now consider properties of the solutions expressed both in the form of (9.16) and also in terms of the reflection functions introduced in (9.18). Considering first the solution (9.x) applied to a semi-infinite medium, namely as $\tau \to \infty$. It follows by assertion that for Therefore (6.26) becomes

We now consider what information about the radiation field in contained in the eigenvectors . Suppose we are deep in a medium somewhere say between depths 100 and 250m as in Fig. 6.5. For a specific incident direction , the intensity field deep in the medium follows as

Here the incident direction is that of the minimum eigenvalue and associated eigenvector corresponding to this minimum eigenvalue, , represents the scattered angular radiance field in the direction . The shape of this distribution is thus invariant with depth and the magnitude decays exponentially according to the factors .

9.4.3 Reflection from a semi-infinite medium

To explore some properties of these reflection functions and their association to the eigenmatrices, consider

$$R(0,\tau) = -M_{++}^{-1} M_{+-}$$
$$= -[\mathbf{u}_+ e^{\Lambda_+\tau}\mathbf{v}_+ - \mathbf{u}_- e^{-\Lambda_+\tau}\mathbf{v}_-]^{-1}[\mathbf{u}_+ e^{\Lambda_+\tau}\mathbf{v}_- - \mathbf{u}_- e^{-\Lambda_+\tau}\mathbf{v}_+]$$

with some effort it can be shown that in the limit as $\tau \to \infty$ then

$$R_\infty = -\mathbf{v}_+^{-1}\mathbf{v}_- \tag{9.21}$$

Figure 9.2 The physical interpretation of the eigenvector $u_{ij}^{\pm}$ representing the asymptotic radiance field deep in a thick medium. Shown are the elements of the eigenvectors (normalized) plotted as a function of μ_i for different assumed values of ϖ_o. The relationship between $u_{ij}^{\pm}$ and μ_i resembles the diffusion pattern shown Fig. 7.6

where R_∞ is an $n \times n$ matrix of bidirectional reflectances of a semi-infinite cloud. This further confirms the general result of Fig. 9.1 and offers an interpretation of R_∞ as representing the asymptotic form of the radiance field.

9.5 Particular solutions

The forced solutions are presented here for the eigenmatrix approach in a general fashion that allows us to consider any τ dependence of the source. However, we will develope the soution for two specific examples, a source of the form

$$\Sigma_{sol}^{\pm}(\mu_i, \tau) = X_{\odot}^{\pm}(\mu_i) e^{-\tau/\mu_{\odot}} \tag{9.22a}$$

where

$$X_{\odot}^{\pm}(\mu_i) = \frac{F_{\odot}}{4\pi} \varpi_o P(-\mu_{\odot}, \mp\mu_i)$$

which corresponds to a source of diffuse solar radiation and a second source of the form

$$\Sigma_{IR}^{\pm}(\mu_i, \tau) = b_o + b_1 \tau \tag{9.22b}$$

which represents that due to thermal emission.

We begin by considering the interaction principle for a layer $x < \tau < y$ which can be written as (see chapter 5, section 5.x)

$$\begin{aligned} I^+(x) &= R(x,y) I^-(x) + T(y,x) I^+(y) + I_p^+(y,x) \\ I^-(y) &= R(y,x) I^+(y) + T(x,y) I^-(x) + I_p^-(x,y) \end{aligned} \tag{9.23}$$

where we are now left with the problem of determining the forced radiance contribution $I_p^{\pm}$

The general matrix form of the radiative transfer equation has the form (e.g. 4.22)

$$\frac{dI}{d\tau} = A\ I + \Sigma \tag{9.24}$$

where

$$A = \begin{pmatrix} -t & r \\ -r & t \end{pmatrix}$$

and

$$\Sigma = \begin{pmatrix} \Sigma^+ \\ \Sigma^- \end{pmatrix}$$

The fundamental solution of this equation is

$$I(\tau) = M(0,\tau)I(0) \tag{9.25}$$

where the matrix $M(0,\tau)$ is given by (9.15). The forced radiances associated with a layer $x < \tau < y$ are obtained as particular solutions of (4.22) of the form

$$\begin{pmatrix} S^+(x,y) \\ S^-(x,y) \end{pmatrix} = \int_x^y \begin{bmatrix} M_{++}(x,s) & M_{+-}(x,s) \\ M_{-+}(x,s) & M_{--}(x,s) \end{bmatrix} \begin{pmatrix} \Sigma^+(s) \\ \Sigma^-(s) \end{pmatrix} ds \tag{9.26}$$

where it is easy to check that

$$\frac{d}{dy} S(x,y) = A\ S(x,y) + \tilde{\Sigma}(y) \tag{9.27}$$

where the proof depends on the fact that for fixed x,

$$\frac{d}{dy} M(x,y) = M(x,y)A$$

which follows by taking the derivitive of each side of (9.25) and reducing using the homogeneous version. The solution to (9.24) then can be writem as

$$\begin{pmatrix} I^+(y) \\ I^-(y) \end{pmatrix} = \begin{bmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{bmatrix} \begin{pmatrix} I^+(x) \\ I^-(x) \end{pmatrix} + \begin{pmatrix} S^+(x,y) \\ S^-(x,y) \end{pmatrix} \tag{9.28}$$

Equating this with (9.23) above, then

$$I_p^+(y,x) = -M_{++}^{-1} S^+(x,y)$$

$$I_p^-(x,y) = -M_{-+} M_{++}^{-1} S^+(x,y) + S^-(x,y) \tag{9.29}$$

Given (9.26) for $S^{\pm}(x,y)$, then the particular solution (that is the 'forced radiance' contribution) follows from (9.29).

We now consider the particular solaution for both types of sources. Consider the solar source in (9.31), then

$$S^+(0,\tau) = \int_0^\tau [\mathbf{u}_+ e^{\Lambda_+ t}\mathbf{v}_+ - \mathbf{u}_- e^{-\Lambda_+ t}\mathbf{v}_-] X_\odot^+ e^{-t/\mu_\odot} dt +$$

$$\int_0^\tau [\mathbf{u}_+ e^{\Lambda_+ t}\mathbf{v}_- - \mathbf{u}_- e^{-\Lambda_+ t}\mathbf{v}_+] X_\odot^- e^{-t/\mu_\odot} dt$$

leading to

$$S^+(0,\tau) = \mathbf{u}_+\mathbf{g}_+[e^{(\Lambda_+ - 1/\mu_\odot)\tau} - E_n][\mathbf{v}_+ X_\odot^+ + \mathbf{v}_- X_\odot^-] -$$

$$\mathbf{u}_-\mathbf{g}_-[e^{(\Lambda_+ + 1/\mu_\odot)\tau} - E_n][\mathbf{v}_- X_\odot^+ + \mathbf{v}_+ X_\odot^-]$$

where $\mathbf{g}_\pm$ are diagonal matrices of $1/(k_j \pm 1/\mu_\odot)$ respectively. (do the same for S^-). This is not in its stable form .In a similar way the IR source follows as

$$S^+(0,\tau) = \int_0^\tau [\mathbf{u}_+ e^{\Lambda_+ t}\mathbf{v}_+ - \mathbf{u}_- e^{-\Lambda_+ t}\mathbf{v}_-](b_o + b_1 t) dt +$$

$$\int_0^\tau [\mathbf{u}_+ e^{\Lambda_+ t}\mathbf{v}_- - \mathbf{u}_- e^{-\Lambda_+ t}\mathbf{v}_+](b_o + b_1 t) dt$$

This integral can be evaluated noting that

$$\int e^{-\mathbf{A}t} dt = -\mathbf{A}^{-1} e^{-\mathbf{A}t}$$

$$\int t e^{-\mathbf{A}t} dt = -\mathbf{A}^{-2}[\mathbf{E} - \mathbf{A}t] e^{-\mathbf{A}t}$$

where again we note $\mathbf{E}$ is the identity matrix. Thus the IR source integral becomes (again not in stable form...

$$S^+(0,\tau) = b_o \mathbf{u}_+ [\Lambda_+ e^{\Lambda_+ t}]_0^\tau (\mathbf{v}_+ + \mathbf{v}_-)$$

$$+ b_1 \mathbf{u}_- [\Lambda_+^{-2}[\mathbf{E} - \Lambda_+ t] e^{-\Lambda_+ t}]_0^\tau (\mathbf{v}_+ + \mathbf{v}_-)$$

9.6 The Greens-function and contribution functions