AT721 Section 8:

Characteristic solutions I: Isotropic scatter

The problem of radiative transfer within an isotropically scattering medium has long historical significance. From the historical perspective, Chandresekhar (1950) demonstrated how the solutions of such problems could be couched in analytic terms involving special functions. In the era before the wide-spread use of computers, this offered obvious practical advantages over the the more general anistropically scattering problems described in the next chapter. The practical advantages, however, did not end there. We have already touched on the concept of similarity which addresses how problems of radiative transfer in anisotropic media can be equated to the isotropically scattering problem via scaling methods and without loss of practical accuracy.

8.1 The theory of scaling

8.1.1 Ball and stick scaling

Consider a phase function of the form

$$P(\cos\Theta) = 2f\delta(0,\Theta) + (1-f)P'(\Theta) \tag{8.1}$$

where $\delta(0,\Theta)$ is a delta function that is only non-zero for scattering angles in the forward direction ($\Theta = 0$, f is some prescribed value defining the magnitude of this forward peak and P' is a simple function of θ, such as given by the Henyey–Greenstein function, a truncated Legendre expansion or even $P' = 1$. The latter is the phase function for isotropic scattering and the phase function in this case resembles that of a ball and stick (Fig. 8.1). Substitution of (8.1) into the radiative transfer equation (which for simplicity and convenience we take the azimuthally averaged form without sources: (– as an exercise you can extend this analysis to include the direct beam term)

$$\mu\frac{dI}{d\tau}(\tau,\mu) = I(\tau,\mu) - \frac{\varpi_o}{2}\int_{-1}^{1}[2f\delta(\mu,\mu') + (1-f)P'(\mu,\mu')]I(\tau,\mu')d\mu'$$

in recalling the properties of δ function integrals, we obtain

$$\mu\frac{dI}{d\tau}(\tau,\mu) = I(\mu)[1-\varpi_o f] - \frac{\varpi_o(1-f)}{2}\int_{-1}^{1}P'(\mu,\mu')I(\tau,\mu')d\mu'$$

Introducing the following quantities

$$d\tilde{\tau} = [1-\varpi_o f]d\tau$$

$$\varpi = \frac{\varpi_o(1-f)}{[1-\varpi_o f]} \tag{8.2}$$

then we obtain an equivalent transfer equation

$$\mu \frac{dI}{d\tilde{\tau}}(\tau,\mu) = I(\tau,\mu) - \frac{\omega}{2}\int_{-1}^{1} P'(\mu,\mu')I(\tau,\mu')d\mu' \tag{8.3}$$

which transforms (scales) one problem as defined by the scattering phase function $P(\cos\Theta)$ to anther as defined by $P'(\cos\Theta)$. A special case is the scaling of an anistropic scattering problem to an isotropic scattering problem $P'(\cos\Theta) = 1$ in which case the anisotropic scattering is represented by a ball and stick model of scattering (Fig. 8.1).

Figure 8.1 The ball and stick phase function.

8.1.2 The δ-M scaling method

Reference: Wiscombe (1977),

So far we have not addressed the issues involved in specifying the factor f. In a more general treatment of scaling, consider the phase function as given via the Legendre expansion

$$P(\cos\Theta) = \sum_{\ell=0}^{M}(2\ell+1)\chi_\ell P_\ell(\cos\Theta) \tag{8.4a}$$

which is truncated at sum level M as indicated by the subscript on the phase function. Functions of the form of (8.1) also also obey the normalization criteria implied under (8.4a) when we use a scaling as follows

$$\chi_\ell^* = \frac{\chi_\ell - f}{1-f} \qquad \text{and} \qquad f = \chi_{M+1} \tag{8.4b}$$

where χ_ℓ^* is used in place of χ_ℓ in (8.3), that is,

$$P'(\cos\Theta) = \sum_{\ell=0}^{M}(2\ell+1)\chi_\ell^* P_\ell(\cos\Theta)$$

is used in (8.1). For the case of isotropic scaling, $M = 0$ and $f = \chi_1 = g$ in which case the scaling parameters of teh isotropic problem become

$$d\tilde{\tau} = [1 - \varpi_o g]d\tau$$

$$\varpi = \frac{\varpi_o(1-g)}{[1-\varpi_o g]} \tag{8.5}$$

Figure 8.2 shows examples of the error of albedo calculated using the $\delta - M$ method of scaling versus a higher order version of the same model (with $M = 50$) as a function of the truncation M.

Expand -show fig of HG phase function expanded and then scaled - do the same for a Mie phase function

8.1.3 Generalized scaling

Ref: Mitrescu and Stephens (2004), JQRST, 86,387-394.

Although the scaling methods described above aim to reduce the complexity of the radiative transfer problem under anisotropic scattering, the choice of scaling factor, f is arbitrary and the number of terms in the expansion required to reproduce a complex phase function is only moderately reduced. Mitrescu and Stephens (2004) introduced a new scaling that both removes the arbitrariness of the specification of f tying it to the amount of scattering in the forward scatter lobe and improves the numerical efficiency over the $\delta - -M$ scaling method.

The method begins with the radiative transfer equation of the form

$$\mu\frac{dI}{d\tau}(\tau,\mu) = I(\tau,\mu) - \frac{\varpi_o}{2}\int_{-1}^{1} P(\mu,\mu')I(\tau,\mu')d\mu'$$

which can be written as

$$\mu\frac{dI}{d\tau}(\mu) = I(\mu) - \frac{\varpi_o}{2}\int_{-1}^{1} P(\mu_s)I(\mu_s)d\mu_s \tag{8.6}$$

where the integral, which physically accomodates the integration over solid angle, has been transformed to one over scattering angle $\cos\theta = \mu_s$. In the process of integrating over solid angle, we introduce a small cone of solid angle defined about the forward scattering direction ($\mu_s = 1$) defined by the parameter α which is the cosine of the scattering angle near the forward direction, then

$$\mu\frac{dI}{d\tau}(\tau,\mu) = I(\tau,\mu) - \frac{\varpi_o}{2}\int_{\alpha}^{1}[P(\mu_s)-P'(\mu_s)]I(\tau,\mu_s)d\mu_s - \frac{\varpi_o}{2}\int_{-1}^{\alpha}P(\mu_s)I(\tau,\mu_s)d\mu_s - \frac{\varpi_o}{2}\int_{\alpha}^{1}P'(\mu_s)I(\tau,\mu_s)d\mu_s$$

where $P'(\mu_s)$ is a function to be defined.

Assuming that the radiance field $I(\mu_s)$ in this forward direction $\alpha < \mu_s < 1$ is uniform enough for the approximation

$$\frac{\varpi_o}{2}\int_{\alpha}^{1}[P(\mu_s)-P'(\mu_s)]I(\tau,\mu_s)d\mu_s \approx \frac{\varpi_o}{2}I(\tau,\mu)\int_{\alpha}^{1}[P(\mu_s)-P'(\mu_s)]d\mu_s \tag{8.7}$$

together with the introduction of the fraction of incident energy scattered in the forward direction

$$f = \frac{1}{2}\int_{\alpha}^{1}[P(\mu_s)-P'(\mu_s)]d\mu_s \tag{8.8}$$

then the radiative transfer equation becomes

$$\mu\frac{dI}{d\tau}(\tau,\mu) = (1-f\varpi_o)I(\tau,\mu) - \frac{\varpi_o}{2}\int_{-1}^{\alpha}P(\mu_s)I(\tau,\mu_s)d\mu_s - \frac{\varpi_o}{2}\int_{\alpha}^{1}P'(\mu_s)I(\tau,\mu_s)d\mu_s \tag{8.9}$$

Figure 8.2 Absolute error in albedo calculated for a scattering layer of optical depth as given, for a solar zenith angle as stated and for $\varpi_o = 1$ for a Mie Phase function (Wiscombe, 1977).

The advantage of this formulation becomes evident in the case of strongly forward asymmetric phase functions. In fact, (8.7) becomes an equality for phase functions of the form (8.1). Since the angle that determines α remains arbitrary, as does the function $P'(\mu_s)$, then f is non-unique as it is for the $\delta - M$ method. Introducing a truncated phase function

$$P''(\mu_s) = P(\alpha)/(1-f), \alpha < \mu_s < 1$$
$$P''(\mu_s) = P(\mu_s)/(1-f), -1 < \mu_s < \alpha \tag{8.10}$$

and set

$$P'(\mu_s) = P(\alpha)/(1-f) \tag{8.11}$$

such that $P''(\mu_s)$ is a continuous function which is helpful when using Legendre expansions for this function. This definition also ensured proper phase function normalization and also provides a way of estimating f solving the following equation obtained by substitution of (8.11) into (8.8)

$$f = \frac{1}{2}\int_{\alpha}^{1} P(\mu_s)d\mu_s - \frac{1}{2}\frac{P(\alpha)}{1-f}(1-\alpha)$$

Show results

8.2 Isotropic scattering: Characteristic solutions (reference Chandrasekhar, p. 18-19)

We write (8.2) in the form

$$\mu\frac{dI}{d\tau} = I - \mathcal{J} \tag{8.12a}$$

where

$$\mathcal{J} = \frac{\varpi_o}{2}\int_{-1}^{1} P\left(\mu,\mu'\right) I\left(\tau,\mu'\right) d\mu'$$

and for isotropic scattering

$$\mathcal{J} = \frac{\varpi_o}{2}\int_{-1}^{1} I\left(\tau,\mu'\right) d\mu' \tag{8.12b}$$

Equations (8.12a) with (8.12b) admit solutions of the form

$$\begin{aligned} I\left(\tau,\mu\right) &= g\left(\mu\right)e^{k\tau} \\ g\left(\mu\right) &= \frac{const}{1-k\mu} \end{aligned} \tag{8.13}$$

which has the same factored form as the assumption we invoked for the approximations discussed in section 7. Here k is a characteristic value (the associated eigenvalue) which can be obtained by substituting (8.13) into (8.12a) and (8.12b) giving

$$\left(1-k\mu\right)g(\mu) = \frac{1}{2}\varpi_o\int_{-1}^{1} g(\mu')d\mu' \tag{8.14}$$

and with (8.13) we obtain from (8.14) the characteristic equation

$$1 = \frac{1}{2}\varpi_o\int_{-1}^{+1}\left(1-k\mu'\right)^{-1}d\mu' = \frac{\varpi_o}{2k}\log\left(\frac{1+k}{1-k}\right) \tag{8.15}$$

Thus k and $(-k)$ are roots of this characteristic equation and it can be shown that for $\varpi_o \leq 1$, there are unique values $(k)^2 < 1$ that satisfy (8.15) as given in Table 8.1.

8.3 Isotropic scattering - the “n - ary” solution (Chand., Chap 3)

8.3.1 Characteristic solution

Introducing the quadrature approximation to the integrals in (8.12b) and (8.12a) lead to a series of equations of the form

$$\mu_i \frac{dI_i}{d\tau} = I_i - \frac{\varpi_o}{2} \sum_{j=-n}^{n} w_j I_j \; (i = \pm 1, \cdots \pm n) \tag{8.16}$$

with solutions of the form

$$I(\tau, \mu_i) = g(\mu_i)\, e^{-k\tau} \tag{8.17}$$

On substitution of (8.17) into (8.18) we obtain

$$g(\mu_i)\;(1 + \mu_i k) = \frac{\varpi_o}{2} \sum_j g(\mu_j)\, w_j$$

from which the characteristic equation

$$1 = \frac{\varpi_o}{2} \sum_{j=-n}^{n} \frac{w_j}{1 + \mu_j k} = \varpi_o \sum_{j=1}^{n} \frac{w_j}{1 - \mu_j^2 k^2} \tag{8.18}$$

is obtained defining $2n$ distinct non zero values of k in pairs $\pm k_j, (j = 1, \cdots n)$. Figure 8.4 shows typical values of k_j for $\varpi_o = 0.95$. Note that for $\varpi_o = 1$, and with $\sum_{j=1}^{n} w_j = 1$, $k^2 = 0$ is a root of (8.18) so there are only $2n - 2$ distinct non zero roots in this case.

Figure 8.4 Typical distribution of eigenvalues for isotropic scattering with a single scattering albedo of 0.95.

The general solution therefore is

$$I\left(\tau, \mu_i\right) = \sum_{j=-n}^{n} \frac{L_j}{1 + k_j \mu_i} e^{-k_j \tau} \tag{8.19a}$$

or in a slightly more expanded form as

$$I\left(\tau, \mu_i\right) = \sum_{j=1}^{n} \frac{L_{-j}}{1 - k_j \mu_i} e^{k_j \tau} + \sum_{j=1}^{n} \frac{L_j}{1 + k_j \mu_i} e^{-k_j \tau} \tag{8.19b}$$

where the $L_j' s$ follow from the given boundary conditions

(b) Particular Solution - Chandresekhar H functions

Consider the source term of the form

$$\mathcal{J} = \frac{\varpi_o}{2} \int_{-1}^{+1} I\left(\tau, \mu'\right) d\mu' + \frac{\varpi_o}{2} F_{\odot} e^{-\tau/\mu_{\odot}} \tag{8.20}$$

Now the solution of (8.4a) given (8.12) can be written as

$$I\left(\tau, \mu\right) = \sum_{j=-n}^{n} \frac{L_j}{1 + k_j \mu_i} e^{-k_j \tau} + I_p(\tau, \mu_i) \tag{8.21a}$$

where

$$I_p\left(\tau, \mu_i\right) = \tilde{\omega}_o \frac{F_{\odot}}{2} q_i e^{-\tau/\mu_{\odot}} \tag{8.21b}$$

On substituting (8.21b) into

$$\mu_i \frac{dI}{d\tau}\left(z, \mu_i\right) = -I\left(z, \mu_i\right) + \frac{\tilde{\omega}_o}{2} \sum_{j=-n}^{n} I\left(z, \mu_j\right) w_j + \frac{\tilde{\omega}_o}{2} F_{\odot} e^{-\tau/\mu_{\odot}}$$

it is easy to verify that q_i obeys

$$q_i(1 + \mu_i/\mu_{\odot}) = \frac{\varpi_o}{2} \sum w_j q_i + 1 \tag{8.22}$$

Chandrasekhar was able to show that

$$q_i = \frac{H(\mu_{\odot}) H(-\mu_i)}{1 + \mu_i/\mu_{\odot}} \tag{8.23a}$$

where

$$H(\mu) = \frac{1}{\mu_1 \mu_2 \cdots \mu_n} \frac{\Pi_{i=1}^{n}(\mu + \mu_i)}{\Pi_{\ell=1}^{n}(1 + k_\ell \mu)} \tag{8.23b}$$

Note also that $H(\mu)$ is a unction of ϖ_o through the dependence of the eigenvalues k_ℓ on ϖ_o.

Van de Hulst (1980) provides various approximations to (8.23b) as given below and in Table 8.2.

Table 8.2 The H function for isotropic scattering and coefficients for its expansion

(1) $\mu_o \to 1$, for a given ϖ_o

$$H(\mu_\odot) = H(1) + k_1(\varpi_o) + k_2(\varpi_o)(1-\mu_\odot)^2 + k_3(\varpi_o)(1-\mu_\odot)^3 \cdots$$

(2) $\varpi_o \to 0$

$$H(\mu_\odot) = 1 + \varpi_o h_1(\mu_\odot)\varpi_o^2(\mu_\odot) + \varpi_o^3 h_3(\mu_\odot) + \cdots$$

(3) $\varpi_o \to 1$

$$H\left(\mu_\odot\right) = H(\mu_\odot)_{\varpi_o=1} + D_1\left(\mu_\odot\right)t + D_2\left(\mu_\odot\right)t^2 + D_3\left(\mu_\odot\right)t^3 \cdots$$

$$t = (1-\varpi_o)^{1/2}$$

8.4 The diffuse reflection for a semi-infinite atmosphere

According to the above discussion, the general solution for an isotropically scattering atmosphere has the form (combining (8.23a) with (8.22) in (8.20))

$$I\left(\tau,\mu_i\right) = \sum_{j=-n}^{n} \frac{L_j}{1+k_j\mu_i}\, e^{-k_j\tau} + \varpi_o \frac{F_\odot}{2} \frac{H(\mu_\odot)H(-\mu_i)}{1+\mu_i/\mu_\odot}\, e^{-\tau/\mu_\odot} \tag{8.24}$$

Consider this solution given the following boundary conditions

- $I(0, -\mu_i) = 0$ (downwelling at top)
- $I(\infty, +\mu_i) = 0$

in which case the boundary condition coefficients become

$$L_j = 0 \qquad j = \pm 1, , \pm n$$

and thus (Chandrasekhar, p 85)

$$I(0, \mu_i) = \frac{\varpi_o F_\odot}{2} \frac{H(\mu_\odot)\ H(\mu_i)}{1 + \mu_i/\mu_\odot} \tag{8.25}$$

which describes the radiance reflected from a semi-infinite atmosphere.

8.4.1 Absorption by deep cloud layers

Reference: Twomey and Bohren, JAS, 1980, 37, 2086-2094.

An example of the application of (8.25) to terrestrial clouds is now considered. This example is presented partly to demonstrate how we might combine radiative transfer principles to derive the bulk radiative properties of an atmospheric layer containing (say) cloud droplets. The cloud layer is considered deep enough to be taken as infinitely thick and the anisotropic scattering by these particles is scaled to isotropic scattering by methods described in section 8.1. The absorption follows directly from calculation of reflection since, in a semi-infinite medium, no light emerges from the bottom of the atmosphere.

Figure 6.4 The geometry for reflection from a semi–infinite medium.

Under these conditions, the reflection from cloud top is given by (8.25), namely

$$I^+(0, \mu_i) = \varpi_o \frac{F_\odot}{2} \frac{H(\mu_o) H(\mu_i)}{1 + \mu_i/\mu_\odot}$$

which on integration yields the reflected flux

$$F^+(0) = 2\pi \sum_{i=1}^{n} w_i \mu_i I^+(0, \mu_i) = \mu_o F_\odot \left[1 - (1 - \tilde{\omega}_o)^{1/2} H(\mu_o)\right] \tag{8.26}$$

where we use

$$\int_0^1 H(\mu) d\mu = \frac{2}{\varpi_o}[1 - (1 - \varpi_o)^{1/2}]$$

(Chandrasekhar, p126). The absorption thus follows as The absorption of a semi-infinite cloud (i.e., thick cloud) is then

$$A = F_{\odot}\mu_o - F^+ = (1-\tilde{\omega}_o)^{1/2}_{\text{isotropic}} H(\mu_o)$$

$$A = \left[\frac{1-\tilde{\omega}_o}{1-g\tilde{\omega}_o}\right]^{1/2}_{\text{anisotropic}} H(\mu_o)$$

where we have invoked the isotropic scaling relationship (8.5), namely

$$(\varpi_o)_{\text{isotropic}} = [(1-g)\varpi_o/(1-\varpi_o g)]_{\text{anisotropic}}$$

A number of studies suggest that the particle absorption

$$1 - \varpi_o \propto a$$

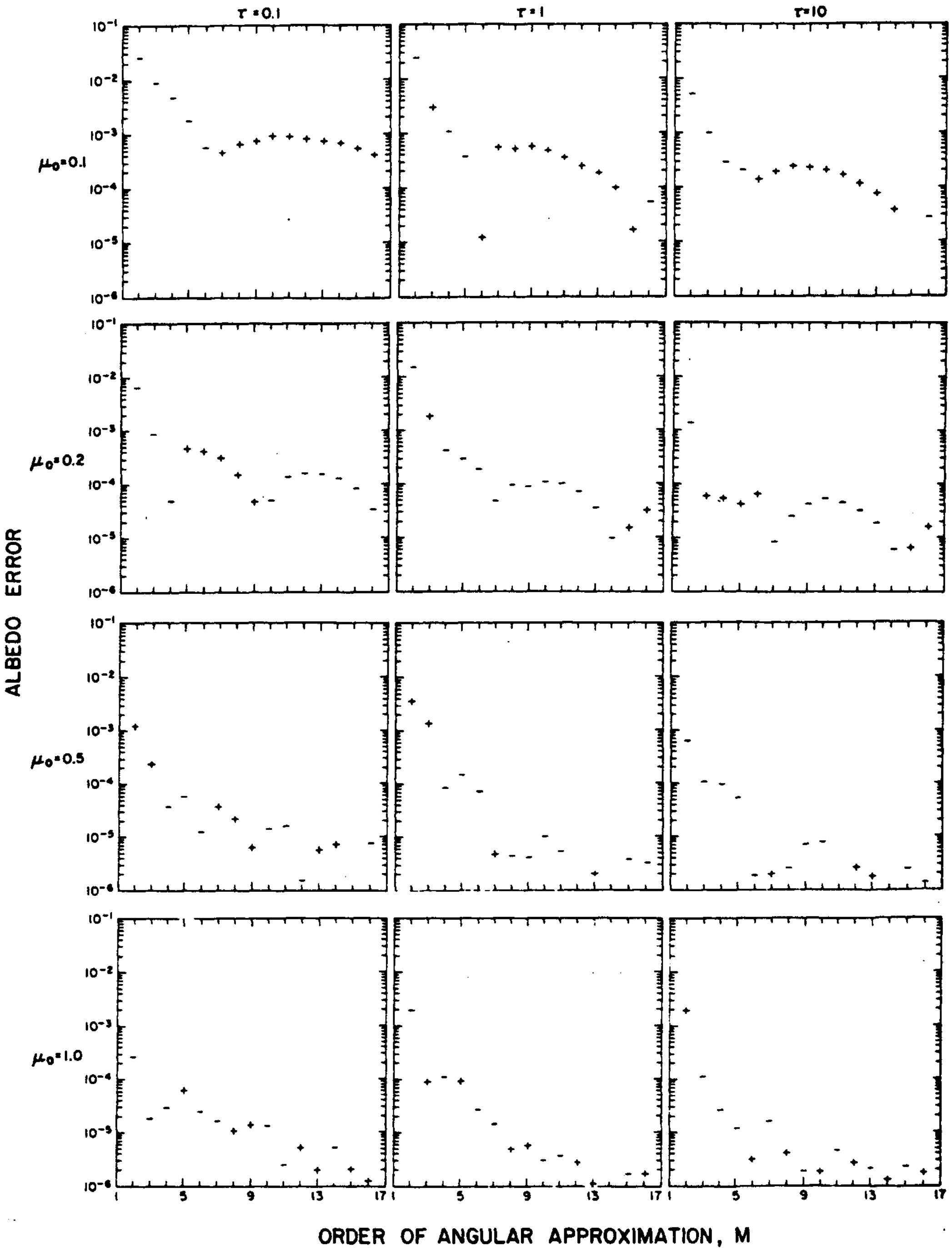
where a is the particle radius, implying then that

$$A \to H(\mu_o)\sqrt{a}$$

Thus from these simple arguments, we would predict that the absorption of a thick anisotropically scattering cloud layer increases with particles size according to an approaximate square root relation as the results of Fig 8.5 indicate.

Figure 8.5 Comparison of results from a Mie–doubling model and the approximate formula based on H functions for two sun elevations (Twomey and Bohren, 1980).

8.5 The diffuse reflection from finite atmospheres: Chandreskhar X/Y functions



τ = 0.1
τ = 1
τ = 10
μ₀ = 0.1
μ₀ = 0.2
μ₀ = 0.5
μ₀ = 1.0
ALBEDO ERROR
ORDER OF ANGULAR APPROXIMATION, M

Fig. 8.2

ω_o	k	ω_o	k
0.0	1.00000	0.8	0.71041
0.2	0.99991	0.9	0.52543
0.3	0.99741	0.925	0.45993
0.4	0.98562	0.95	0.37948
0.5	0.95750	0.975	0.27111
0.6	0.90733	1.000	0.0
0.7	0.82864	–	–

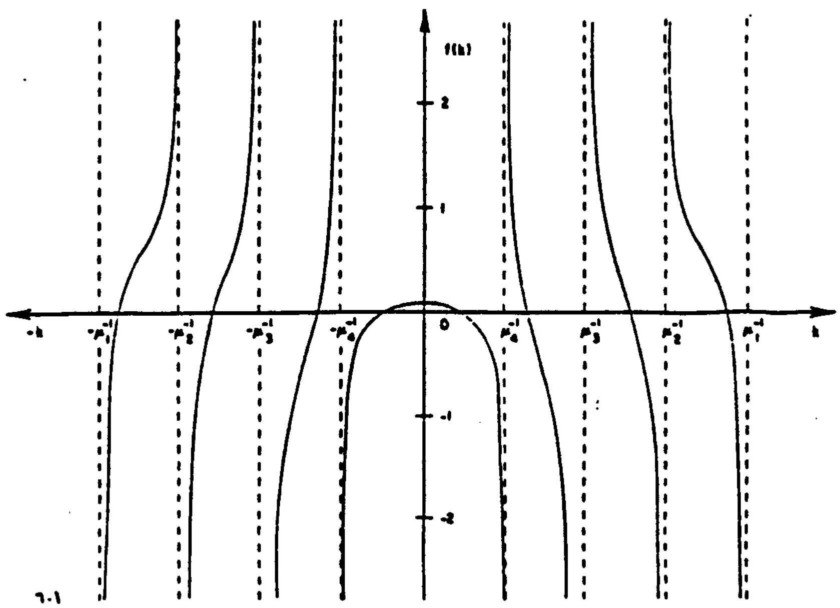
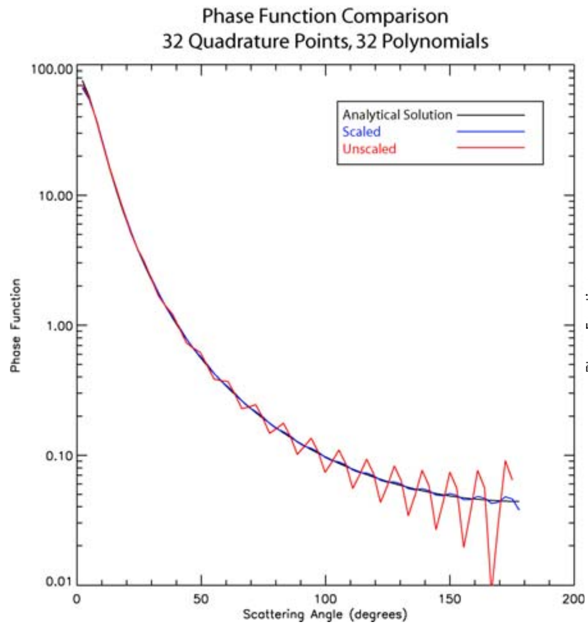
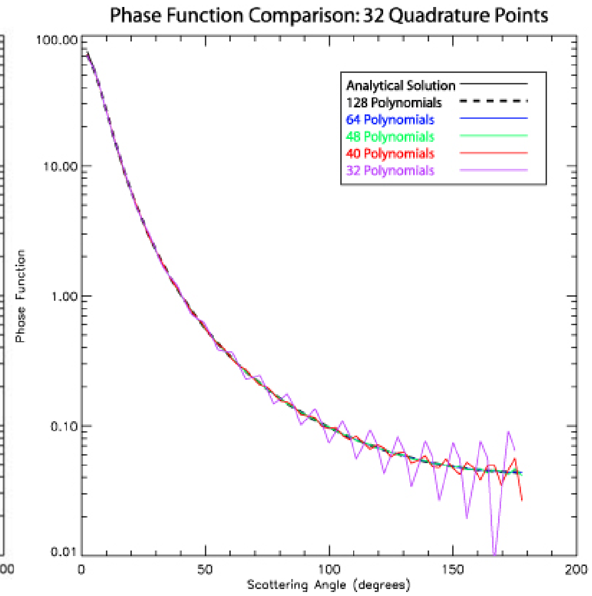


Fig. 8.3


Phase Function Comparison
32 Quadrature Points, 32 Polynomials
Analytical Solution
Scaled
Unscaled
Phase Function
Scattering Angle (degrees)
100.00
10.00
1.00
0.10
0.01
0
50
100
150
200


Phase Function Comparison: 32 Quadrature Points
Analytical Solution
128 Polynomials
64 Polynomials
48 Polynomials
40 Polynomials
32 Polynomials
Phase Function
Scattering Angle (degrees)
100.00
10.00
1.00
0.10
0.01
0
50
100
150
200

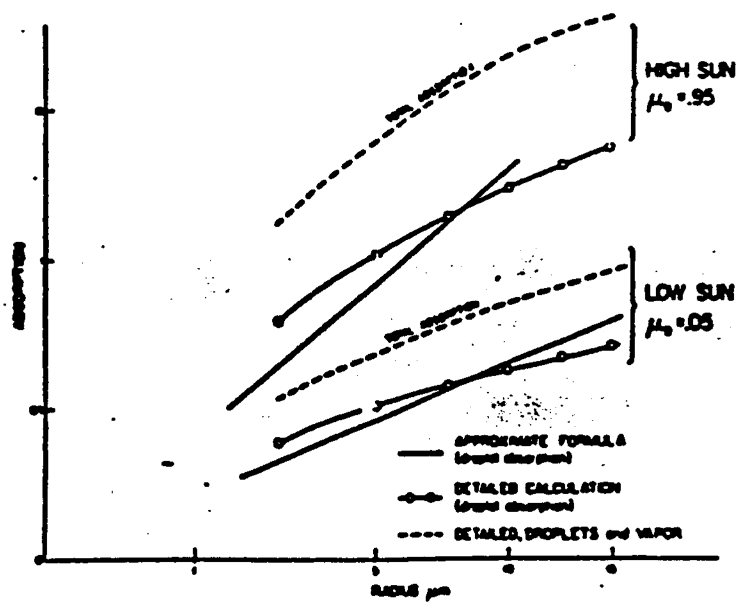

HIGH SUN
μ_0 =.95
LOW SUN
μ_0 =.05
DETAILED CALCULATION

Fig. 8.5