# AT721 Section 5:

# The Interaction Principle

It might appear surprising that the interaction principle receives little discussion in many radiative transfer studies yet the principle is the foundation for the solution to most of the radiative transfer problems we encounter in the atmospheric sciences. For example, adding a reflecting surface below a scattering layer involves both application of the interaction principle and subsequent steps that are fundamental to solving the most general of multiple scattering problems.

The principle of interaction embodies the general linear nature of the interactions between radiation and matter. Before introducing this principle here, and before solving any particular problem, we will first introduce properties that describe how a surface both reflects and transmits radiation in different directions making use of the matrix formulations developed in the previous chapter.

## 5.1 (Bi–directional) Reflection and Transmission Functions

To begin with, consider the thin slab of material illustrated in Fig. 5.1. The surface may be the top of a cloud, the top of the atmosphere, the ground, or any other "surface" capable of both reflecting and transmitting radiation. If the surface is illuminated by radiation with an intensity $I_i$ along the direction $\hat{\xi}'$ contained in an element of solid angle of $d\Omega(\hat{\xi}')$, then the

$$\text{incident flux density} = I_i(\vec{r}, \hat{\xi}')d\Omega(\hat{\xi}')$$

The flux density reflected from the surface into the set of directions $d\Omega(\hat{\xi})$ along the direction $\hat{\xi}$ produces a

$$\text{response flux } = I_r(\vec{r}, \hat{\xi})d\Omega(\hat{\xi})$$

and the ratio of these fluxes defines a generalized reflection function

$$r(\vec{r}, \hat{\xi}, \hat{\xi}') = \frac{I_r(\vec{r}, \hat{\xi})d\Omega(\hat{\xi})}{I_i(\vec{r}, \hat{\xi}')d\Omega(\hat{\xi}')}$$

Suppose that the flux density emerging from the surface is measured by an instrument looking at the surface. If this instrument collects radiation within a field of view defined by a small solid angle $d\Omega_{fov}$, then it follows that the measured intensity is

$$I_r(\vec{r}, \hat{\xi}) = \frac{1}{d\Omega_{fov}} r(\vec{r}, \hat{\xi}, \hat{\xi}')I_i(\vec{r}, \hat{\xi}')d\Omega(\hat{\xi}')$$

If the flow is such that it opposes the incoming flow of radiation, namely if $\hat{\xi} \cdot \hat{\xi}' < 0$, then the radiation is said to be reflected, and we define

$$\mathcal{R}(\vec{r}, \hat{\xi}, \hat{\xi}') = \frac{1}{d\Omega_{fov}} r(\vec{r}, \hat{\xi}, \hat{\xi}')$$

**Figure 5.1** A thin slab of a surface showing the diffuse reflection and transmission separately.

as the bidirectional reflection function such that the reflected intensity becomes

$$I_r(\vec{r}, \hat{\xi}) = \mathcal{R}(\vec{r}, \hat{\xi}, \hat{\xi}')I(\vec{r}, \hat{\xi}')d\Omega(\hat{\xi}') \tag{5.1}$$

This reflection function applies equally to a surface that is infinitely deep, as might be assumed in the example of a land surface, or to a thin slab of material in a vacuum such as illustrated in Fig. 5.1.

Using entirely analogous arguments, when the flow from the surface is observed in the same sense as the flow onto the surface, i.e. $\hat{\xi} \cdot \hat{\xi}' > 0$, then the radiation is said to be transmitted through the slab such that

$$\mathcal{T}(\vec{r}, \hat{\xi}, \hat{\xi}') = \frac{1}{d\Omega_{fov}} \frac{I_t(\vec{r}, \hat{\xi})d\Omega(\hat{\xi})}{I_i(\vec{r}, \hat{\xi}')d\Omega(\hat{\xi}')}$$

defines the bidirectional transmittance function. The intensity of radiation transmitted through the slab is therefore

$$I_t(\vec{r}, \hat{\xi}) = \mathcal{T}(\vec{r}, \hat{\xi}, \hat{\xi}')I(\vec{r}, \hat{\xi}')d\Omega(\hat{\xi}') \tag{5.2}$$

## 5.2 The Interaction Principle

Equations (5.1) and (5.2) set the stage for writing down the interaction principle. In so doing, consider now a surface illuminated by multiple sources of radiation where $I(\vec{r},\hat{\xi}'_k)$ is the intensity of the $k$th source below and $I(\vec{r},\hat{\xi}'_j)$ is the intensity of the $j$th source above, then the radiation emerging from the surface along the direction $\hat{\xi}$ is

$$\begin{aligned} I(\vec{r},\hat{\xi}) = & \sum_{\text{no of j sources}} \mathcal{R}(\hat{\xi},\hat{\xi}'_j) I_i(\hat{\xi}'_j) d\Omega(\hat{\xi}'_j) \\ & + \sum_{\text{no of k sources}} \mathcal{T}(\hat{\xi},\hat{\xi}'_k) I_i(\hat{\xi}'_k) d\Omega(\hat{\xi}'_k) \end{aligned} \tag{5.3}$$

or alternatively for a continuum of such sources;

$$\begin{aligned} I(\vec{r},\hat{\xi}) = & \int \mathcal{R}(\hat{\xi},\hat{\xi}') I(\hat{\xi}') d\Omega(\hat{\xi}') \\ & + \int_{\mathcal{T}} (\hat{\xi},\hat{\xi}") I(\hat{\xi}") d\Omega(\hat{\xi}") \end{aligned} \tag{5.4}$$

This is a statement of the principle of interaction – the resultant intensity observed leaving the surface is just a simple superposition of reflected and transmitted intensities from all impinging sources. This expression can be written in a convenient way if we consider a set of n discrete incident and n distinct response directions $(\pm\hat{\xi}_1 \pm \hat{\xi}_2 \ldots \pm \hat{\xi}_n)$ and write

$$I^{\pm} = \Big[ I(\pm\hat{\xi}_1), I(\pm\hat{\xi}_2) \ldots I(\pm\hat{\xi}_n) \Big]^t$$

as a column vector of intensities such that $I^+$ is the set of $n$ intensities flowing along the $n$ directions $\hat{\xi}_1 \ldots \hat{\xi}_n$ and $I^-$ is the set of $n$ intensities flowing along the $n$ directions $-\hat{\xi}_1 \ldots -\hat{\xi}_n$ .

Amply this to develop the expression in polar co-ord form

Using the interaction principle expressed in the form of (5.4), together with the vectorial representation of the intensities, we can now determine the intensities emerging from a scattering layer as shown in Fig. 5.2. For convenience, the scattering layer is considered to extend from optical depth levels $a$ to $b$ and is illuminated from above by a diffuse source of intensity characterized by the vector $I^-(a)$ . $R_g$ is the matrix of bidirectional reflection functions of the surface and examples of the matrix form of this function are given in the next section. The following $R(a,b), R(b,a), T(a,b)$ and $T(b,a)$ represent matrices of bidirectional reflection and transmission functions of a layer of atmosphere such that

$$\begin{aligned} I^-(b) &= T(a,b)I^-(a) + R(b,a)I^+(b)(+Q(a,b)) \\ I^+(a) &= R(a,b)I^-(a) + T(b,a)I^+(b)(+Q(b,a)) \end{aligned} \tag{5.5}$$

is an alternative way of writing (5.4). We have also generalized the interaction principle to include sources $Q(a,b), Q(b,a)$ within the layer.

## 5.3 Examples of surface reflection functions

The interaction principle can also be caste in terms of different types of reflection and transmission functions that can be defined according to the nature of the radiometric quantities of interest. In many

**Figure 5.2** The interaction Principle

problems of atmospheric radiation, it is often convenient to introduce the surface reflection in terms of an albedo quantity usually defined with respect to hemispheric fluxes. Consider the reflected component of (5.4)

$$I(\vec{r}, \hat{\xi}) = \int \mathcal{R}(\hat{\xi}, \hat{\xi}') I(\hat{\xi}') d\Omega(\hat{\xi}')$$

and write this in the form

$$I(\vec{r}, \hat{\xi}) = \int \frac{\mathcal{R}(\hat{\xi}, \hat{\xi}')}{\mid \hat{\xi}' \cdot \hat{k} \mid} \mid \hat{\xi}' \cdot \hat{k} \mid I_i(\hat{\xi}') d\Omega(\hat{\xi}') \tag{5.6}$$

where the factor $\mid \hat{\xi}' \cdot \hat{k} \mid$ is the cosine of the angle between the direction of incident radiation and the normal to a reference surface taken to be a horizontal surface. In this context,

$$F_{\odot} = \int I_i(\hat{\xi}') \mid \hat{\xi}' \cdot \hat{k} \mid d\Omega(\hat{\xi}')$$

is the incident hemispheric flux. Equation (5.6) leads naturally to a new reflection function

$$\frac{\mathcal{R}(\hat{\xi},\hat{\xi}')}{\mid \hat{\xi}' \cdot \hat{k} \mid}$$

from which we derive the reflected radiances given incident radiances or the incident flux as input.

### 5.3.1 *The example of a Lambertian reflecting surface*

By far the most used surface type in radiative transfer studies is the Lambertian surface, although more for its mathematical convenience than for reality. A Lambertian surface reflects isotropically with a surface albedo $\alpha$. That is

$$\frac{\mathcal{R}(\hat{\xi},\hat{\xi}')}{\mid \hat{\xi}' \cdot \hat{k} \mid} = \frac{\alpha}{\pi} = \text{constant}$$

and the reflected radiance is then

$$I(\vec{r},\hat{\xi}) = \frac{\alpha}{\pi}\int I(\vec{r},\hat{\xi}') \mid \hat{\xi}' \cdot \hat{k} \mid d\Omega(\hat{\xi}') = \frac{\alpha}{\pi}F_{\odot}$$

or in polar co-ordinates

$$I(\vec{r},\mu,\phi) = \frac{\alpha}{\pi}\int_0^{2\pi} d\phi' \int_0^1 I(\vec{r},\mu',\phi')\mu' d\mu'$$

We can also write this expression in the interaction form of (5.4)

$$I^+(b) = R_g I^-(b) \tag{5.7}$$

where $R_g$ is the matrix of bi-directional reflection functions of the surface. In discrete quadrature form, this reflection matrix takes the form

$$R_g^m = \frac{2\alpha\delta_m}{\sum_{i=1}^n w_i\mu_i}\begin{pmatrix} w_1\mu_1 & \dots & w_n\mu_n \\ \vdots & \ddots & \vdots \\ w_1\mu_1 & \dots & w_n\mu_n \end{pmatrix} \tag{5.8}$$

where $w_{1,\dots,n}$ and $\mu_{1,\dots,n}$ are the quadrature weights and abscissae and $\delta_m = 1$ for $m = 0$ and zero otherwise.

This reflection matrix characterizes the reflection of diffuse radiation. In the case of incident light composed of both diffuse and direct radiation, the reflection is

$$I^+(b) = R_g^m I^-(b) + \alpha\frac{\mu_\odot F_\odot}{\pi}e^{-\tau/\mu_\odot}\mathbf{U} \tag{5.9}$$

where $\mathbf{U}$ is a vector of unity for $m = 0$ and zero otherwise.

### 5.3.2 *A more general example*

here we consider surface reflection functions that depend only on the scattering angle and not on the specific angles of incidence or reflectance. The reflection from such a surface has the form

$$I^+(b) = R_g^m I^-(b) + \frac{\mu_\odot F_\odot}{\pi}e^{-\tau/\mu_\odot}\gamma \tag{5.10}$$

where

$$R_g^m = \frac{1}{\sum_{i=1}^n w_i\mu_i} \begin{pmatrix} w_1\mu_1 \sum_\ell^N \chi_\ell Y_\ell^m(-\mu_1)Y_\ell^m(\mu_1) & \dots & w_n\mu_n \sum_\ell^N \chi_\ell Y_\ell^m(-\mu_1)Y_\ell^m(\mu_n) \\ \vdots & \ddots & \vdots \\ w_1\mu_1 \sum_\ell^N \chi_\ell Y_\ell^m(-\mu_n)Y_\ell^m(\mu_1) & \dots & w_n\mu_n \sum_\ell^N \chi_\ell Y_\ell^m(-\mu_n)Y_\ell^m(\mu_n) \end{pmatrix}$$

and

$$\gamma = (2-\delta_m) \begin{pmatrix} \sum_\ell^N \chi_\ell Y_\ell^m(mza)Y_\ell^m(\mu_1) \\ \vdots \\ \sum_\ell^N \chi_\ell Y_\ell^m(mza)Y_\ell^m(\mu_n) \end{pmatrix}$$

and where $\chi_\ell$ are the coefficients of expansion of the bi-directional surface reflection function.

#### *5.3.3 Other examples*

#### *5.3.4 The Cox and Munk model*

### 5.4 The radiative transfer equation as a statement of the interaction principle

So far we have introduced the radiative transfer equation and are interested in its solution applied to many different problems. As we will discover later, solution of this equation can be carried out using repeated applications of the interaction principle in the form of relationships like that (5.4). We might now ask what is the relation between the principles introduced above and the radiative transfer equation for, once answered, we will be able to construct the reflection, transmission and source functions and use the interaction principle to develop a solution.

We begin by considering a thin layer of atmosphere of optical depth $\Delta\tau$ (Fig.5.3) and re-write the portion of (5.4) relevant to $I_\nu^+$ in an approximate finite difference form (with obvious notational simplifications)

$$\frac{I^+(\Delta\tau) - I^+(0)}{\Delta\tau} = tI^+(\Delta\tau/2) - rI^-(\Delta\tau/2) - \Sigma^+(0,\Delta\tau)$$

where we note $r$ and $t$ are $n \times n$ matrices and $I^\pm$ and $\Sigma^+$ are column vectors of length n. With re-arrangement

$$I^+(0) = [E - t\Delta\tau]I^+(\Delta\tau) + r\Delta\tau I^-(0) + \Sigma^+(0,\Delta\tau)\Delta\tau \tag{5.11}$$

where for simplicity

$$I^+(\Delta\tau/2) \approx I^+(\Delta\tau)$$

$$I^-(\Delta\tau/2) \approx I^-(0)$$

Equation (5.11) is merely a statement of the interaction principle and by matching it to (5.4) term by term, we obtain

$$\begin{aligned} T(\Delta\tau) &= T(b,a) = E - t\Delta\tau \\ R(\Delta\tau) &= R(a,b) = r\Delta\tau \\ Q^+(\Delta\tau) &= Q(b,a) = \Sigma^+\Delta\tau \end{aligned} \tag{5.12}$$

The implication of this simple exercise is as follows;

•It establishes the connection between the interaction principle and the radiative transfer equation - the latter being merely a differential form of the former
•It provides a way of determining the 'global' reflection and transmission functions, R(a,b), and T(b,a) and global source function $Q(b,a)$, from the intrinsic scattering and absorption properties of the medium, namely $r$ and $t$ and $\Sigma^+$.
•Once determined, complex problems in radiative transfer can be solved

### 5.4.1 Establishing the initial step-size

## 5.5 Example applications of the interaction principle

### 5.5.1 Example 1: A lower reflecting surface

In this example we are interested in determining how much radiation is reflected from the top of a layer that overlies a reflecting surface given the properties of that surface and the reflection and transmission functions of the atmosphere. For simplicity, we will at first assume the atmosphere to be sourceless. From (5.4) we write

$$I^-(b) = T(a,b)I^-(a) + R(b,a)I^+(b)$$
$$I^+(a) = R(a,b)I^-(a) + T(b,a)I^+(b)$$

for the atmosphere. It is also possible to write an interaction statement for surface reflection as

$$I^+(b) = R_g I^-(b)$$

Re–arrangement of these equations yields

$$I^+(a) = [R(a,b) + T(b,a)R_g[1 - R(b,a)R_g]^{-1}T(a,b)]I^-(a) \tag{5.13}$$

where the first term of the right–hand side is the reflection of the incident light by scattering from the layer $a \to b$. The second term is the contribution to the reflected light by multiple scattering between $a \to b$ and the surface. The important factor in this expression is the factor

$$P = [E - R(b,a)R_g]^{-1}$$

which we refer to as the propagator factor. A graphical depiction of the multiple scattering between the surface and atmospheric layer in Fig. 5.4 offers a clear interpretation of this factor.

**Modify this analysis to include sources from a to b**.

### 5.5.2 Example 2: Composition relationships for two arbitrary sourceless layers

This is a variant of Example 1 where now a transparent layer of the atmosphere is added ($b \to c$) below the original layer that extends from $a \to b$. We require the intensities emerging from the combined layers, namely we require $I^+(a)$ and $I^-(c)$ given incident intensities. The problem as posed is presented in Fig. 5.5

We start by writing the interaction principle for the two layers individually:

$$I^-(b) = T(a,b)I^-(a) + R(b,a)I^+(b) \qquad I$$

$$
\begin{aligned}
I^+(a) &= R(a,b)I^-(a) + T(b,a)I^+(b) \qquad III \\
I^-(c) &= T(b,c)I^-(b) + R(c,b)I^+(c) \qquad II \\
I^+(b) &= R(b,c)I^-(b) + T(c,b)I^+(c) \qquad IV
\end{aligned} \tag{5.14}
$$

where the Roman numerals refer to the given 'principles of invariance' introduced by Chandrasekhar (1950) and identified with the same numbering convention as given in that reference (p. xx). Similarly we can write the interaction principle for the combined layer $a \to c$

$$
\begin{aligned}
I^-(c) &= T(a,c)I^-(a) + R(c,a)I^+(c) \qquad I \\
I^+(a) &= R(a,c)I^-(b) + T(c,a)I^+(c) \qquad III
\end{aligned} \tag{5.15}
$$

In eliminating the internal radiances $I^{\pm}(b)$ from the four relationships above and then by subsequently matching the resulting expression term-by-term with the interaction principle of the combined system (5.x), it follows that

$$
\begin{aligned}
T(a,c) &= T(b,c)[E - R(b,a)R(b,c)]^{-1}T(a,b) \\
T(c,a) &= T(b,a)[E - R(b,c)R(b,a)]^{-1}T(c,b) \\
R(a,c) &= R(a,b) + T(b,a)[E - R(b,c)R(b,a)]^{-1}R(b,c)T(a,b) \\
R(c,a) &= R(a,b) + T(b,c)[E - R(b,c)R(b,a)]^{-1}R(b,c)T(c,b)
\end{aligned} \tag{5.16}
$$

### 5.5.3 Example 3: The Composition relationships for two arbitrary layers with sources

Show how we get

$$
Q^{\pm}_{n+1} = T_n P_n [R_n Q^{\mp}_n + Q^{\pm}_n] + Q^{\pm}_n \tag{5.17}
$$

## 5.6 Further composition methods: homogeneous layers and the idea of doubling

The composition relations (5.16) are known as imbedding relationships. For a vertically homogeneous (i.e. the properties of layer $a \to b$ are the same as those in layer $b \to c$ then we can simply employ these relationships in a cyclic manner to build of the global functions for very thick layers. A strategy for doing so is illustrated in Fig. 5.6 and is referred to as the method of doubling. We start with the R, T and Q functions determined for the optically thin layer $\Delta\tau$ using relationships like those introduced in (5.12). Wit the first application of the imbedding relationships (5.16) and (5.17), we obtain $R(2\Delta\tau), T(2\Delta\tau)$ and $Q(2\Delta\tau)$ and in $p$ cycles of this application, we build these functions for a layer $2^p\Delta\tau$ thick. Thus the doubling procedure can be expressed by re-writing (5.16) and (5.17) in the following simple algorithmic form:

$$
\begin{aligned}
R_{n+1} &= R_n + T_n P_n R_n T_n \\
T_{n+1} &= T_n P_n T_n \\
Q^{\pm}_{n+1} &= T_n P_n [R_n Q^{\mp}_n + Q^{\pm}_n] + Q^{\pm}_n
\end{aligned}
$$

where

$$
P_n = [E - R_n R_n]^{-1}
$$

is the propagator factor. Interpretation of (5.16) is such that $n = 0, 1, 2, p-1$ refers to the cycle number initialized with $R_0 = R(\Delta\tau), T_0 = T(\Delta\tau)$ and $P_0 = P(\Delta\tau)$.

### 5.6.1 Choosing $\Delta\tau$

There are a few criteria that need to be kept in mind when choosing a value of $\Delta\tau$. Naturally, $\Delta\tau$ has to be sufficiently small that single scatter is predominant within the layer $\Delta\tau$ thick. But it also must be chosen such

$$T_{i,i}(\Delta\tau) = 1 - t_{i,i,max}\Delta\tau > 0$$

or

$$t_{i,i,max}\Delta\tau < 1$$

which follows from (5.12) and the desired to keep the elements of T(i,i) positive definite. Generally, this usually implies that $dt < 0.001$. Thus suppose we are interested in building up reflection functions for a layer of optical thickness $\tau = 10$, then for

$$2^p\Delta\tau = 10$$

and $dt < 0.001$ we obtain p¿10.

### 5.6.2 Extension to include specific vertically varying source terms

## 5.7 Further composition methods: inhomogeneous layers and the idea of adding

## 5.8 Ricatti methods

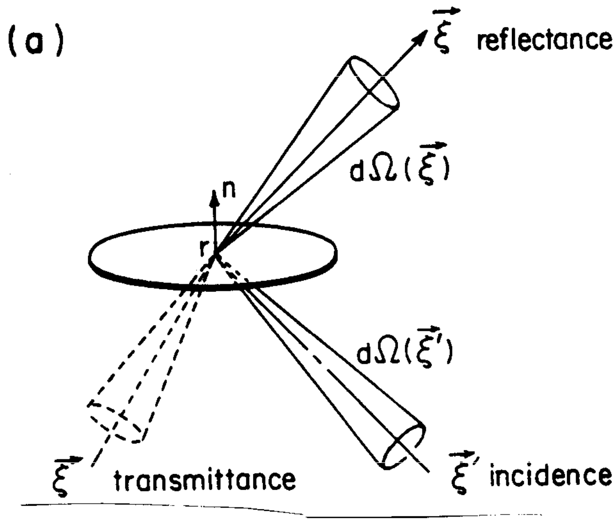
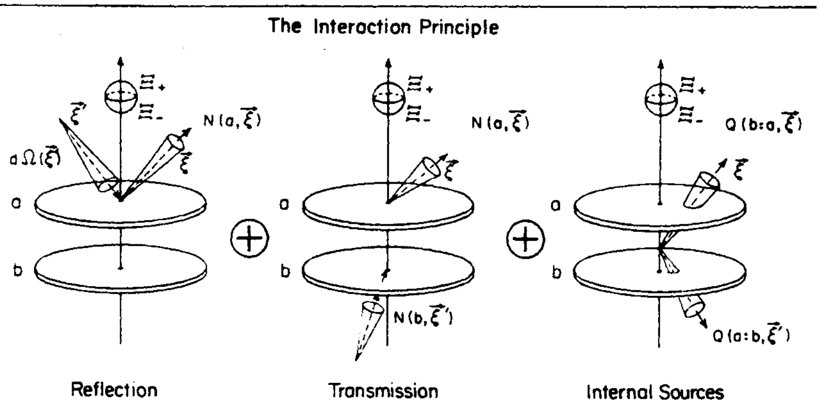



Fig 5.1




$$N(a,\vec{\xi}) = R(a:b,\vec{\xi},\vec{\xi}')N(a,\vec{\xi}')d\Omega(\vec{\xi}') + T(b:a,\vec{\xi},\vec{\xi}')N(b,\vec{\xi}')d\Omega(\vec{\xi}') + Q(a:b,\vec{\xi})$$

Fig 5.2

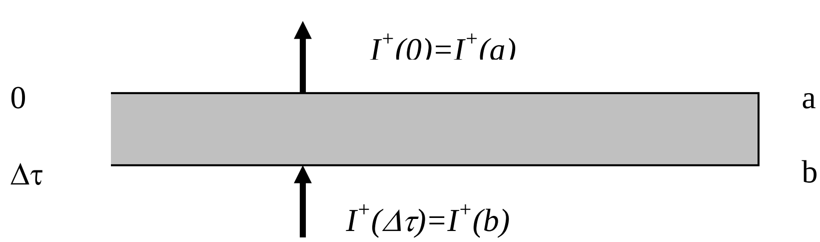



Fig 5.3

R(a,b) +T(b,a)RgT(a,b) + T(b,a)RgR(b,a)RgT(a,b) + ....

Fig 5.4

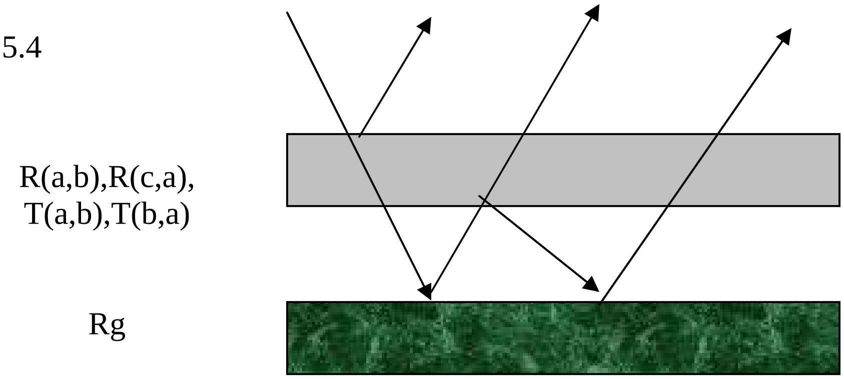



Fig 5.5

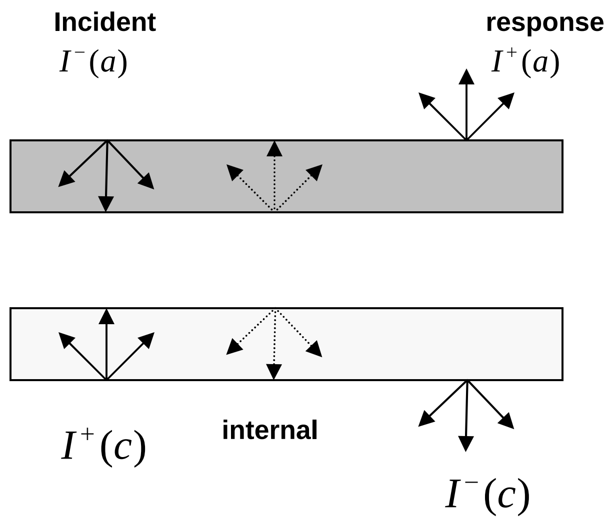



Fig 5.6