AT721 Section 2:

The Radiative Transfer Equation

Although we can find a number of different forms of radiative transfer equation in the literature, each are merely an expression or approximation of energy conservation.

2.2 Volumetric Energy balance

Perhaps the most instructive way to consider radiative transfer is in terms of the energy balance of a small, elemental volume of 'medium' in which processes occur to both created and destroy radiation. The mathematical details of these processes is addressed later.

Figure 2.1 caption here.

For now, imagine an elemental volume of cross-sectional area dA located at a point in space as defined by the general position vector $\vec{r}$ (Fig. 2.1). Suppose the quantity:

$$dI_\nu(\vec{r}, \hat{\xi})dAd\Omega(\hat{\xi})d\nu dt \tag{2.1}$$

represents the difference in the radiative energy after a beam has passed through the volume crossing the surfaces $dA(\vec{r})$ and $dA(\vec{r} + d\vec{r})$ at each end of the volume element (Fig. 2.1) after traversing a path ds. From the definition of spectral intensity (1.5), this change can be interpreted as the energy contained within a time interval dt centered about t and in the frequency interval $d\nu$ centered about ν, and further confined to an element of solid angle $d\Omega$ oriented in the direction $\hat{\xi}$ which defines the direction the radiation is propagating. Suppose $\mathcal{S}_\nu(\vec{r}, \hat{\xi})$ represents the net gain per unit path length of radiative energy as the beam traverses this volume element along ds. Then the quantity

$$\mathcal{S}_\nu(\vec{r}, \hat{\xi})dsdAd\Omega(\hat{\xi})d\nu dt \tag{2.2}$$

is the net elemental gain (i.e. sum of losses and gains) of radiative energy by the beam confined as it traverses a cylindrical element of volume $dAds$. Equating (2.1) and (2.2) produces

$$\frac{dI_\nu(\vec{r},\hat{\xi})}{ds} = \mathcal{S}_\nu(\vec{r},\hat{\xi}). \tag{2.3}$$

We can further write (2.3) as1

$$\frac{dI_\nu(\vec{r},\hat{\xi})}{ds} = \hat{\xi}\cdot\nabla I_\nu(\vec{r},\hat{\xi}) = \mathcal{S}_\nu(\vec{r},\hat{\xi}). \tag{2.4}$$

Equation (2.4) is a very general statement of energy balance expressed in differential form and, so far, is independent of any coordinate system. In the cartesian system

$$\hat{\xi}\cdot\nabla = \hat{\xi}\cdot[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}]$$

$$= \cos\phi\sin\theta\frac{\partial}{\partial x} + \sin\phi\sin\theta\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z}$$

and the general differential form of radiative transfer becomes

$$\cos\phi\sin\theta\frac{\partial I_\nu(\vec{r},\hat{\xi})}{\partial x} + \sin\phi\sin\theta\frac{\partial I_\nu(\vec{r},\hat{\xi})}{\partial y} + \cos\theta\frac{\partial I_\nu(\vec{r},\hat{\xi})}{\partial z} = \mathcal{S}_\nu(\vec{r},\hat{\xi}) \tag{2.5}$$

Many problems in radiative transfer, and the majority of those considered in this book deal with media that are stratified and horizontally uniform such that

$$\frac{\partial I_\nu(\vec{r},\hat{\xi})}{\partial x}, \frac{\partial I_\nu(\vec{r},\hat{\xi})}{\partial y} = 0$$

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1 If c is the speed of propagation of radiation within the medium, the distance ds traversed by the beam is $ds = cdt$. Then (2.3) becomes:

$$\frac{1}{c}\frac{dI_\nu(\vec{r},\hat{\xi})}{dt} = \mathcal{S}_\nu$$

where $\frac{d}{dt}$ is the material derivative:

$$\frac{dI_\nu(\vec{r},\hat{\xi},t)}{dt} = \frac{\partial I_\nu}{\partial t} + c\hat{\xi}\cdot\nabla I_\nu.$$

Therefore it follows that

$$\frac{dI_\nu(\vec{r},\hat{\xi},t)}{ds} = \frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{\xi}\cdot\nabla I_\nu.$$

and since $c \gg 0$,

$$\frac{dI_\nu(\vec{r},\hat{\xi},t)}{ds} \approx \hat{\xi}\cdot\nabla I_\nu.$$

and

$$\cos\theta \frac{dI_\nu(z,\mu,\phi)}{dz} = \mathcal{S}_\nu(z,\mu,\phi) \tag{2.6}$$

which is referred to as the plane-parallel form of radiative transfer equation

2.3 Sources and sinks of radiation

To complete the derivation of transfer equation, the specific components of $\mathcal{S}_\nu$ are now defined. In so doing, we restrict the discussion to media that absorb, emit and scatter radiation and formulate these with respect to small volume of 'medium' illustrated in Fig. 2.1. Two processes remove radiation from this volume as the beam traverses it and the combined losses are referred to as extinction. One process is the absorption of radiation and the second is that due to scattering of some fraction out of the traversing beam. By contrast, processes that add radiation to the beam are referred to as emission and there are two forms of emission (Fig. 2.2b). One is the real thermal emission usually considered under condition of local thermodynamic equilibrium and the is a 'virtual emission' arising from the scattering of all radiation that impinges on the volume from all directions into the direction of the beam $\hat{\xi}$.

2.3.1 Extinction

Extinction is expressed as a change in intensity dI_ν when a beam propagates along a path of length ds. This change is empirically related to the incident intensity of the radiation via Lambert's law of extinction

$$dI_\nu = -\sigma_{ext} I_\nu ds \tag{2.7}$$

where σ_{ext} is the proportionality constant known as the extinction coefficient. This extinction may occur as a result of scattering by particles or molecules in the atmosphere, by absorption by particles and molecules in the atmosphere or by a combination of both (although the molecules that scatter radiation are, on the whole, different from the molecules that absorb radiation). Thus we can write

$$\sigma_{ext} = \sigma_{sca} + \kappa_\nu \tag{2.8}$$

where σ_{sca} and κ_ν are the scattering and absorption coefficients. In equating (2.7) to (2.x), we obtain

$$\mathcal{S}_\nu(\vec{r},\hat{\xi}) = -\sigma_{ext}(\vec{r}) I_\nu(\vec{r},\hat{\xi}) \tag{2.9}$$

2.3.2 Thermal Emission

For the majority of radiative transfer problems encountered in the atmospheric sciences consider that the emission of radiation occurs under the assumption of local thermodynamic equilibrium (LTE). In media that are in a state of LTE, the emission from any small volume element within the medium is characterized by its local temperature $T(\vec{r})$. In such situations

$$\mathcal{S}_\nu(\vec{r},\hat{\xi}) = -\kappa_\nu(\vec{r}) B(T(\vec{r})) \tag{2.10}$$

where the emission is specified in terms of the Planck black body function $B(T(\vec{r}))$.

2.3.3 Diffuse Scattering as a virtual source of Radiation

Photons flowing along a given direction are removed by single scattering. This process is contained within eh extinction formula (2.7). However, these photons can also reappear again along that same direction when scattered a multiple of times. In fact, many of the scattering media of interest to studies of the atmosphere are multiple scattering media, that is media containing a sufficient number of scatterers that photons traversing it are likely to be scattered more than once. The whiteness and brightness of clouds is a result of multiple scattering as is the non-uniform color of the clear sky. Reflection of visible and microwave radiation from various surfaces is also largely influenced by multiple scattering. Multiple scattering is relevant to a myriad of topics and the mathematical description of how we account for it occupies much of the first half of this book.

Figure 2.2 Add fig from LITE Miller and Stephens, 1999.

For now we are concerned only indirectly with multiple scattering in the sense that we wish to account for its effects on a beam of monochromatic radiation as it flows through a volume taken to be small enough that only single scattered photons emerge from it along a direction $\hat{\xi}$ defined by the beam that illuminates it. The incremental increase in intensity along the direction $\hat{\xi}$ due to the scattering of a beam incident from the direction $\hat{\xi}'$ is, by virtue of the definition of the phase function,

$$\delta I_\nu(\vec{r},\hat{\xi}) = \sigma_{sca}(\vec{r}) ds \frac{P_\nu(\vec{r},\hat{\xi}\cdot\hat{\xi}')}{4\pi} I_\nu(\vec{r},\hat{\xi}') d\Omega(\hat{\xi}') \qquad (2.10).$$

where the wavelength dependence on all quantities is understood. The total contribution to $I_\nu(\vec{r},\hat{\xi})$ by scattering of the complete diffuse field surrounding the volume is given by the integral of (2.10), namely

$$dI_\nu(\vec{r},\hat{\xi}) = \sigma_{sca}(\vec{r}) ds \int_{4\pi} \frac{P_\nu(\vec{r},\hat{\xi}\cdot\hat{\xi}')}{4\pi} I_\nu(\vec{r},\hat{\xi}') d\Omega(\hat{\xi}') \qquad (2.11).$$

which leads to the following definition

$$J(\vec{r},\hat{\xi}) = \varpi_o(\vec{r}) \int_{4\pi} \frac{P_\nu(\vec{r},\hat{\xi}\cdot\hat{\xi}')}{4\pi} I_\nu(\vec{r},\hat{\xi}') d\Omega(\hat{\xi}') \qquad (2.11).$$

Figure 2.3 Geometry for scattering of diffuse light. $\hat{\xi}$ the unit vector that defines the direction of the flow and $\vec{r}$ is the vector that specifies the position of the volume element relative to an origin point.

where $\varpi_o = \frac{\sigma_{sca}}{\sigma_{ext}} \leq 1$ is the single scattering albedo. Thus

$$\mathcal{S}_\nu(\vec{r},\hat{\xi}) = \sigma_{sca} J(\vec{r},\hat{\xi})$$

We also make note here of an important property that derives from the definition of the phase function,

$$\frac{1}{4\pi}\int_{4\pi} P_\nu(\vec{r},\hat{\xi}\cdot\hat{\xi}')d\Omega(\hat{\xi}') = 1 \tag{2.12}$$

which merely states that the amount of energy totally scattered out of a beam is equal to the amount of energy removed from the forward direction by scattering.

2.4 The equation of transfer revisited

We are now able to bring together the three components of $\mathcal{S}(\vec{r},\hat{\xi})$ to arrive at the following general form of radiative transfer equation for an absorbing, emitting and scattering medium;

$$\frac{dI_\nu(\vec{r},\hat{\xi})}{ds} = -\sigma_{ext}(\vec{r})[I_\nu(\vec{r},\hat{\xi}) - J(\vec{r},\hat{\xi})] + \kappa_\nu B(T(\vec{r})) \tag{2.13}$$

or

$$\frac{dI_\nu(\vec{r},\hat{\xi})}{ds} = -\sigma_{ext}(\vec{r})I_\nu(\vec{r},\hat{\xi}) + \sigma_{sca}(\vec{r})\int_{4\pi}\frac{P_\nu(\vec{r},\hat{\xi}\cdot\hat{\xi}')}{4\pi}I_\nu(\vec{r},\hat{\xi}')d\Omega(\hat{\xi}') + \kappa_\nu B(T(\vec{r})) \tag{2.14}$$

Consider now a number of special cases:

- *The sourceless medium* , $J(\vec{r},\hat{\xi}), B(T(\vec{r})) = 0$. In this case,

$$\frac{dI_\nu(\vec{r},\hat{\xi})}{ds} = -\sigma_{ext}(\vec{r})I_\nu(\vec{r},\hat{\xi}) \tag{2.15}$$

and the solution to this equation is straightforward and expresses Beer's law, namely

$$I_\nu(\vec{r}(s''),\hat{\xi}) = I_\nu(\vec{r}(s'),\hat{\xi})\exp[-\int_{\vec{r}(s')}^{\vec{r}(s'')}\sigma_{ext}(\vec{r}(s')ds'] \tag{2.16}$$

where the exponential factor is the transmission function.

- *Absorbing-emitting atmosphere*, $\varpi_o = 0,\ J(\vec{r},\hat{\xi}) = 0, \sigma_{ext}(\vec{r}) = \kappa_\nu(\vec{r})$.
- *The plane parallel equation*
  In cartesian coordinates, the plane-parallel of (2.14) becomes

$$\mu\frac{dI_\nu(z,\mu,\phi)}{dz} = -\sigma_{ext}(z)I_\nu(z,\mu,\phi) + \frac{\sigma_{sca}(z)}{4\pi}\int_{\phi'=0}^{2\pi}\int_{\mu'=-1}^{1}P_\nu(z,\mu,\phi,\mu',\phi')I_\nu(z,\mu',\phi'd\mu'd\phi'$$

$$+\kappa_\nu(z)\mathcal{B}(T(z)) \tag{2.17}$$

Many radiative transfer problems adopt a (vertical) coordinate defined by the quantity

$$d\tau = -\sigma_{ext}dz \tag{2.18}$$

which is referred to as the optical depth. Of note is the '-' sign which implies the optical depth increases *downwards* from the top of the atmosphere opposite to z which increases *upwards* from the surface. The consequence of this follows when combining (2.18) into (2.17), to produce

$$\mu\frac{\mathrm{I}_\nu(\tau,\mu,\phi)}{d\tau} = I_\nu(\tau,\mu,\phi) - \frac{\varpi_o(\tau)}{4\pi}\int_{\phi'=0}^{2\pi}\int_{\mu'=-1}^{1}P_\nu(\tau,\mu,\phi,\mu',\phi')I_\nu(\tau,\mu',\phi'd\mu'd\phi' - [1-\varpi_o]\mathcal{B}(T(\tau)) \tag{2.19}$$

where a (subtle) sign change to the individual terms on the right hand side of (2.19) now appears.

- *Diffuse-direct radiation;* It is desirable to develop some way of dealing separately with collimated sources of radiation, such as the direct radiation from the sun. For these problems it proves to be practical to separate the radiation along the direction of this source (we call this the direct beam) from the radiation in all other directions (which we refer to as the diffuse field as it arises from scattering of the direct beam). In making this distinction, then we need to consider two distinct virtual emission sources, one arising from scattering of the diffuse field and a second due to scattering of the direct beam. Consider the intensity field as follows

$$I = I^0 + I^1 + I^2 + \cdots I^n + \cdots = \sum_n I^n \tag{2.20}$$

where for short–hand the $\vec{r}$, and $\hat{\xi}$ dependencies have been dropped. here we have decomposed the intensity field into a sum of all orders of scattering and there is much more discussion on this approach in chapter x. We write this decomposition in the form

$$I = I^0 + I^* \tag{2.21}$$

where I^* is the total diffuse intensity $I^* = \sum_{n=1} I^n$ and I^0 is the unscattered radiation (i.e. the radiation we associate with penetration of a collimated source into the medium. The radiative transfer equation that defines the variation of I^0 is simply Beers law which can be written in differential form

$$\mu_\odot\frac{dI^0}{dz} = -\sigma_{ext}I^0$$

for the beam travelling along the direction $(\mu_\odot, \phi_\odot)$ and has the solution

$$I^0(z) = I_\odot \exp[-\tau/\mu_\odot] \tag{2.22}$$

where $I_\odot$ is the source incident at the top of the medium. Introducing (2.22) in (2.19) leads to the following equation for $\mu \neq \mu_\odot$

$$\mu\frac{dI^*(z,\mu,\phi)}{dz} = -\sigma_{ext}(z)I^*(z,\mu,\phi) + \frac{\sigma_{sca}(z)}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(z,\mu,\phi,\mu',\phi')[I^*(z,\mu',\phi') + I^0(z,\mu',\phi')]d\mu' d\phi'$$
$$+\kappa_\nu(z)B(T(z)) \tag{2.23}$$

where we make specific provision for scattering of I^0 from the direction (μ', ϕ') into the direction (μ, ϕ). For a collimated (or near collimated source), the direction (μ', ϕ') is constrained about a small sub–set of the Ω–space such that the term

$$\frac{\sigma_{sca}(z)}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(z,\mu,\phi,\mu',\phi')I^0(z,\mu',\phi')d\mu' d\phi' = \frac{\sigma_{sca}(z)}{4\pi}\int_{4\pi} P(z,\hat{\xi},\hat{\xi}')I^0(z,\hat{\xi}')d\Omega(\hat{\xi}')$$
$$= \frac{\sigma_{sca}(z)}{4\pi}\int_{\Omega_\odot} P(z,\hat{\xi},\hat{\xi}')I^0(z,\hat{\xi}')d\Omega(\hat{\xi}')$$

where $\Omega_\odot$ is a small but finite solid angle characteristic of the near-collimated beam. Since $\Omega_\odot$ is small, it follows that

$$\frac{\sigma_{sca}(z)}{4\pi}\int_{\Omega_\odot} P(z,\hat{\xi},\hat{\xi}')I_\odot(z,\hat{\xi}')d\Omega(\hat{\xi}') \approx \frac{I^0(z)\Omega_\odot}{4\pi}\sigma_{sca}(z)P(z,\hat{\xi},\hat{\xi}_\odot)$$

and from (2.x) it follows that the **source of diffuse intensity due to scattering of a collimated source** at the level specified by τ is

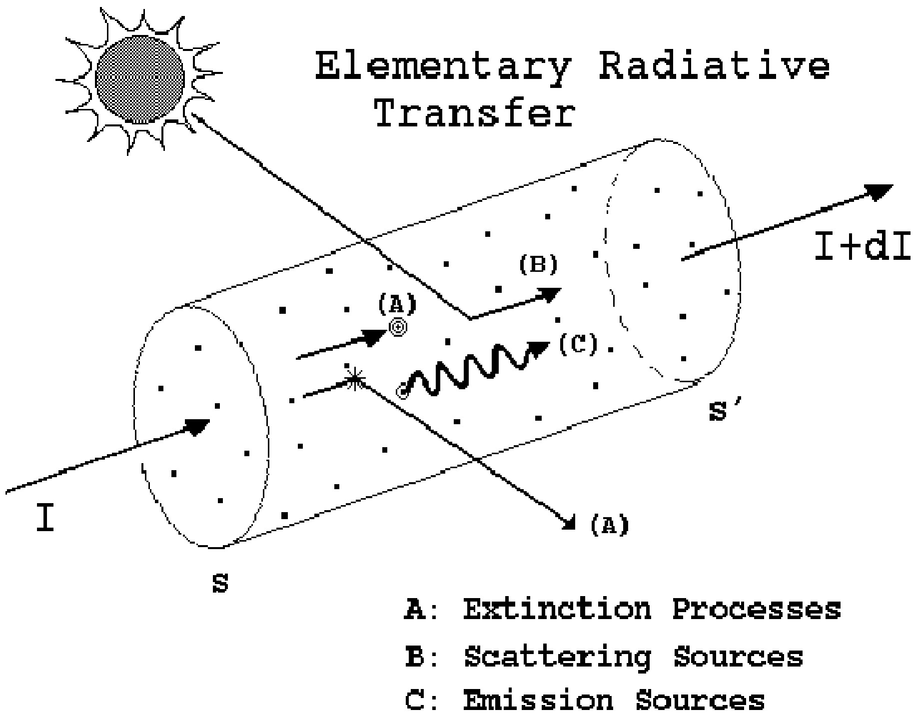
$$\frac{F_\odot}{4\pi}\sigma_{sca}(z)P(z,\hat{\xi},\hat{\xi}_\odot)\exp[-\tau/\mu_\odot]$$

where $F_\odot = I_\odot\Omega_\odot$ is the flux of radiation incident at cloud top. Combining this with (2.22) leads to

$$\mu\frac{dI^*(z,\mu,\phi)}{dz} = -\sigma_{ext}(z)I^*(z,\mu,\phi) + \frac{\sigma_{sca}(z)}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(z,\mu,\phi,\mu',\phi')I^*(z,\mu',\phi')d\mu' d\phi'$$
$$+\frac{F_\odot}{4\pi}\sigma_{sca}(z)P(z,\mu,\phi,\mu_\odot,\phi_\odot)\exp[-\tau/\mu_\odot] + \kappa_\nu(z)\mathcal{B}(T(z)) \tag{2.24}$$

Hereafter we will omit the superscript '*' and take it to be understood that I in equations of the form of (2.24) refers to the diffuse intensity. The alternative form of (2.24) is

$$\mu\frac{dI(\tau,\mu,\phi)}{d\tau} = I(\tau,\mu,\phi) - \frac{\varpi_o(\tau)}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(\tau,\mu,\phi,\mu',\phi')I(\tau,\mu',\phi')d\mu' d\phi'$$
$$-\frac{F_\odot}{4\pi}\varpi_o(\tau)P(\tau,\mu,\phi,\mu_\odot,\phi_\odot)\exp[-\tau/\mu_\odot] - [1-\varpi_o](\tau)\mathcal{B}(T(\tau)) \tag{2.25}$$


Elementary Radiative Transfer
(A)
(B)
(C)
I+dI
S'
I
S
(A)
A: Extinction Processes
B: Scattering Sources
C: Emission Sources

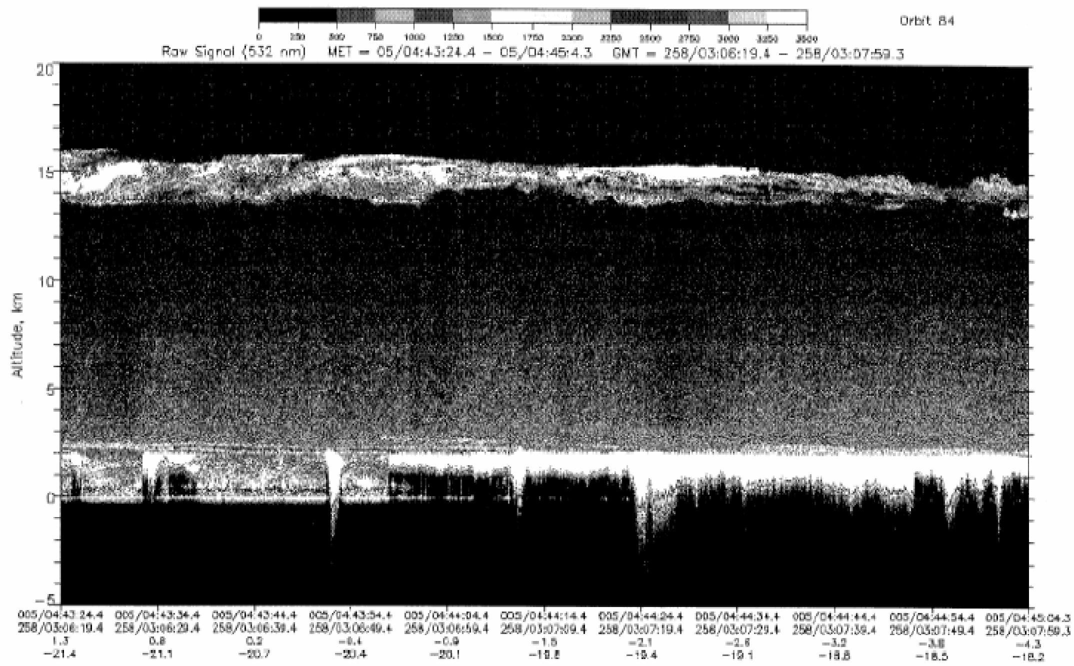


Figure 16. Raw count return from orbit 084 showing pulse extension effects in lower level clouds but not in 15-km cirrus (image courtesy of NASA Langley Research Center).

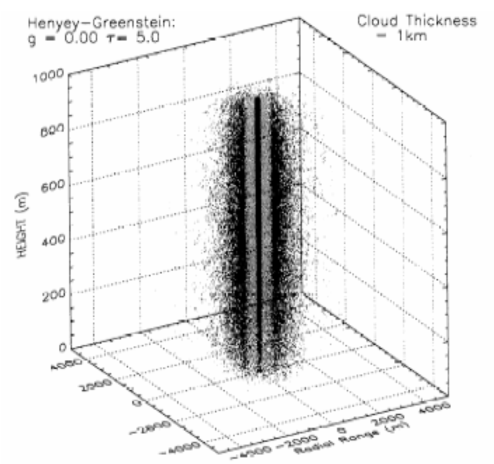


Figure 4. Scattering events for $\tau = 5.0$, isotropic.

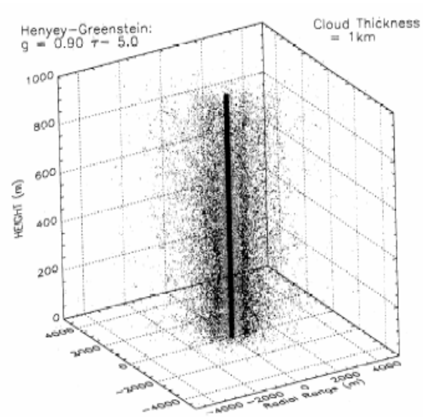
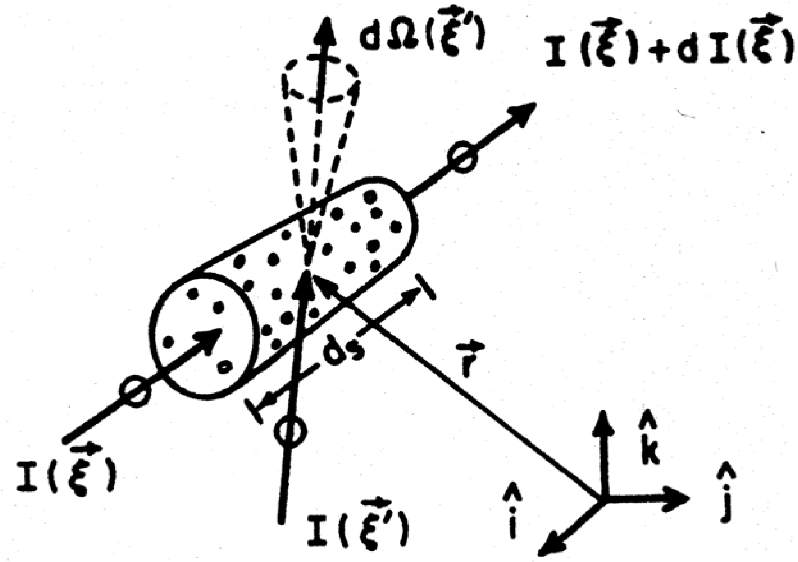


Figure 5. Scattering events for $\tau = 5.0, g = 0.90$, Henyey-Greenstein.



dΩ(ξ')
I(ξ)+dI(ξ)
ds
r
k
j
i
I(ξ)
I(ξ')