AT721 Section 1:

Elementary Concepts

1. Introduction

Transport theory that accounts for the major radiation processes in the Earth's atmosphere (scattering, absorption and emission) including multiple scattering is one specific case of a more general class of problems (Table 1.1). The radiative transfer equation is equivalent to Boltzmann's equation used as the basis of the kinetic theory of gases and in the transport of neutrons. The same theory is used in a host of applications – such as underwater visibility, marine biology, optics of paint, papers, and photographic emulsions, radiation propagation in planets, stars and galaxies and in the optics of biological material such as blood, viruses and others.

1.2. Scope

This book deals with both forward and inverse problems of radiative transfer as typically encountered in the Earth–atmosphere sciences. The stage for these topics, and the scope of this book, can be broadly set in the context of the following expression:

$$y = F(x, b) + \varepsilon_y \tag{1.1}$$

where x is the vector representing the physical properties of the atmosphere (temperature, moisture, particle information, etc.,) that we wish to infer from the measurements y which is a vector generally of radiometric quantities, ε_y is the measurement error including all instrument factors (noise and calibration uncertainties for example), and F is the forward function connecting the measurements to the physical properties of the atmosphere. For the topics of this book, F describes the processes of radiative transfer and formally represents the solution of the radiative transfer equation. b is a vector of other parameters that define F and will be assumed to be known. For example, b might represent appropriate spectroscopic information on gaseous absorption, refractive index properties or other information about particle scattering, etc.

In general the function F is imperfectly known to us and approximations have to be introduced to establish it. These approximations depend on the types of radiative transfer problems that characterize our problem. For instance, F may be the solution to (multiple) scattering processes when visible radiances are measured or F might be defined by emission and absorption in the far IR. The first portion of this book deals largely with the formulation of the radiative transfer equation for these different problems and the corresponding methods of solution.

The second half of the book is concerned with the inversion of F, in various approximate forms, to arrive at an estimate of x. In practice most problems of these inversion problems are ill-posed and significantly affected by errors in f as well as errors in the measurements themselves. This places a certain onus on evaluating the solutions to the radiative transfer equation. Typically we overcome these problems by introducing some form of constraint, usually in the form of a constraint on x via a priori information. Practically all inversions, simple or complex, use constraints either explicitly or implicitly. An example of the explicit use of constaints is in sounding retrievals in which profile information is used as an initial guess in retrievals of temperature or moisture. This error source is not a major concern if it is known that a priori data does not propagate into the final retrieval. Unfortunately, it is generally not understood just how much this sort of information is retained in the finalretrieval.

1.3 Geometry and direction

Before developing the details of radiative transfer and its solutions, we must first consider the relevant quantity around which the theory is constructed. As we will soon see this quantity is the spectral intensity or spectral radiance and it has at its cores, a geometric construct. The intensity defines the flow of energy along a specific collection of directions and before considering this quantity in more detail, we must first establish what we mean by a direction and how we represent integration over direction. To do so requires the introduction of a reference coordinate system. Most generally, this is the Cartesian system specified by three orthogonal axes x, y, and z and their corresponding unit vectors $\hat{\imath}, \hat{\jmath}$, and $\hat{k}$ respectively. Two examples of such a system in common use in Earth sciences are shown in Fig. 1.1. We refer to either one as a *terrestrial frame of reference*. What distinguishes these two systems from each other is the way the x and z axes are anchored relative to each other. In both cases, the z axis is parallel to the local vertical but in one case z increases in a direction opposite to the direction of $\hat{k}$. Another special property of the two reference frames shown is that the x axis is aligned so the $x - z$ plane contains the sun. This is a special situation and such a frame is known as a *sun–based frame of reference.*

A general reference point within a Cartesian frame of reference may be indicted by the position vector $\vec{r}$ such that

$$\vec{r} = (x, y, z),$$

where (x, y, z) defines the coordinates of the tip of this vector. We can also define a direction vector in terms of a general unit position vector $(\hat{\xi})$ which has its base at the origin and tip at the point (a, b, c) where this point lies on the unit sphere that surrounds the origin. In this case $\sqrt{a^2 + b^2 + c^2} = 1$. The unit direction vector may also be defined in terms of a general point (x, y, z) by

$$\hat{\xi} = \vec{r} / \mid r \mid = (x, y, z) \,/\, \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}.$$

A more trigonometrical interpretation of the direction vector follows by considering Fig. 1.2a. A point (a, b, c) on the unit sphere, has the coordinates

$$\begin{aligned} a &= \hat{\xi} \cdot \hat{\imath} = \cos\phi \sin\theta \\ b &= \hat{\xi} \cdot \hat{\jmath} = \sin\phi \sin\theta \\ c &= \hat{\xi} \cdot \hat{k} = \cos\theta = \mu, \end{aligned}$$

where θ is the zenith angle and ϕ is the azimuth angle. The latter, in this case, is measured positive counter clockwise from the x axis. Since $\vec{\xi} = (a, b, c)$, then

$$\hat{\xi} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) \tag{1.2}$$

and it often proves convenient to replace $\vec{\xi}$ with the angle pair (θ, ϕ) where the latter means (1.1). We will also use μ for $\cos\theta$ throughout the book and will hereafter interchangeably use $\hat{\xi}$,(θ, ϕ) or (μ, ϕ) to represent a direction.

1.3.1 The example of the Scattering angle

Figure 1.1 Two sun–based terrestrial frames of reference commonly used in studies of the scattering of radiation in the earth's atmosphere.

Figure 1.2 (a) Angle and direction definitions defined with respect to a unit sphere. (b) Scattering geometry and the scattering angle on the unit sphere.

The scattering angle Θ is an important parameter in the mathematical description of the scattering process. This angle is defined as the angle between the direction of incident radiation $\hat{\xi}$ and the direction of scattered radiation $\hat{\xi}'$,

$$\cos\Theta = \xi \cdot \hat{\xi}'. \tag{1.3}$$

which can also be schematically represented on a unit sphere as shown in Fig. 1.2b. It follows from (1.2) and (1.3) that Θ can be stated in terms of the two pairs of angles (μ', ϕ') and (μ, ϕ)

$$\cos\Theta = \mu\mu' + \left(1-\mu^2\right)^{\frac{1}{2}}\left(1-\mu'^2\right)^{\frac{1}{2}}\cos\left(\phi'-\phi\right) \tag{1.4}$$

and, on occasions, it will prove convenient to replace $\cos\Theta$ with the notation (μ, μ', ϕ, ϕ').

1.1.1 The example of ..

add other examples

1.4 Solid angle and hemispheric integrals

Practically all radiative transfer problems, and those dealing with scattering in particular, require some type of angular integration. This requires the notion of solid angle and a convenient way to think about both these angular integrals, and the concept of solid angle specifically, is to imagine that a point source of light is located at the center of our unit sphere and that there exists a small hole of area A on its surface allowing light to flow through it. This light is contained in a small cone of directions which is represented by the solid angle element

$$\Omega = \text{ area of opening on unit sphere } \Xi,$$

that is

$$\Omega = \frac{A}{r^2}$$

where Ξ symbolically represents the unit sphere and where r is the radius of this unit sphere. The area of the opening is then

$$r^2\Omega = r^2\int_A d\Omega = r^2\int\int_A dadb = r^2\int\int_A sin\theta d\theta d\phi$$

where an r^2 factor is dropped since $r = 1$ for the unit sphere. Therefore, the solid angle element $d\Omega$, which has units of *steradian*,is related to θ and ϕ according to

$$d\Omega = \sin\theta d\theta d\phi.$$

Example 1: The solid angle of sphere, hemisphere

Example 2: The solid angle of the sun

The solid angle of a spherical segment is

$$\Omega = \int_{\theta_1}^{\theta_2} sin\theta d\theta \int_0^{2\pi} d\phi = 2\pi[\cos\theta_1 - \cos\theta_2]$$

The solid angle of a spherical cap defined by the angle θ is

$$\Omega = \int_0^{2\pi} d\phi \int_0^{\theta} sin\theta d\theta = 2\pi[1 - \cos\theta]$$

and for small $\theta_o, \rightarrow \cos\theta_o \rightarrow 1 - \theta_o^2/2 + \ldots$ and

$$\Omega_\odot \approx \pi\theta_o^2$$

which defines the he solid angle of the sun. To a good approximation $\theta_o \approx r_\odot/R_S$ and thus

$$\Omega_\odot \approx \pi\left(\frac{0.7\times 10^6}{1.510^8}\right)^2 \approx 0.684\times 10^{-4}$$

1.5 Spectral Intensity and angular moments

Figure 1.3 Geometry and symbols used in the definition of intensity.

The monochromatic intensity (or radiance) is the fundamental quantity around which radiative transfer is constructed. To define this quantity, consider an element of surface dA located at $\vec{r}$ and oriented with a unit normal $\hat{n}$ as illustrated in Fig 1.3. Let dE_ν be the amount of radiative energy in the frequency interval between ν and $\nu+d\nu$, confined to an element of solid angle $d\Omega$ around the direction of propagation $\hat{\xi}$ streaming through the element of surface dA during the time interval $t, t+dt$. Let θ be the polar angle between the normal $\hat{n}$ and the direction of propagation $\hat{\xi}$. The monochromatic intensity $I_\nu(\vec{r},\hat{\xi})$ as follows:

$$dE_\nu = I_\nu(\vec{r},\hat{\xi})dA\cos\theta d\Omega d\nu dt \tag{1.5}$$

or

$$I_\nu(\vec{r},\hat{\xi}) = \frac{dE_\nu}{dA\cos\theta d\Omega d\nu dt} \tag{1.6}$$

where $dA\cos\theta = dA\hat{\xi}\cdot\hat{n}$ is the projection of the surface dA normal to the direction of flow $\hat{\xi}$. In words, the spectral intensity is the amount of energy streaming through unit area perpendicular to the direction of propagation $\hat{\xi}$, per unit solid angle around the direction $\hat{\xi}$, per unit frequency about the frequency ν, and per unit time about the time t. The units of spectral radiance hereafter are $Wm^{-2}.ster^{-1}.f^{-1}$ where f here symbolically represents a measure of the spectral region of interest, either as a unit of frequency (Hz), a unit of wavelength (m) or a unit of wavenumber (m^{-1}). Hereafter it is taken to be understood that the spectral intensity is a function of this frequency measure as indicated by the subscript ν and this dependence will often be dropped for convenience. Also, the time dependence ...

1.5.1 Angular integrals

Most radiative transfer problems require some integration of the intensity $I_\nu(\vec{r},\hat{\xi})$, over some set of directions defined by some solid angle $\Omega(\hat{\xi})$ which may be taken as centered about $\hat{\xi}$. The integral

$$h(\hat{\xi}) = \int_{\Omega(\hat{\xi})} I_\nu(\vec{r},\hat{\xi}) d\Omega(\hat{\xi})$$

defines a flux quantity (units of $Wm^{-2}.f^{-1}$). For the case that $\Omega = 4\pi$, then the quantity

$$\bar{I_\nu} = \frac{1}{4\pi}\int_{\Omega(\hat{\xi})} I_\nu(\vec{r},\hat{\xi}) d\Omega(\hat{\xi}) = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{-1}^{1} I_\nu(\vec{r},\mu,\phi) d\mu \tag{1.6}$$

is referred to as the mean intensity.

Consider now an integral performed with respect to some fixed surface element dA oriented in such a way that the unit vector $\hat{n}$ is the normal to the surface. According to (1.5)

$$dE_\nu = I_\nu(\vec{r},\hat{\xi}',t) dA \hat{n}\cdot\hat{\xi} d\Omega(\hat{\xi}) d\nu dt$$

is the energy that flows through the surface dA. The total energy confined to solid angle $\Omega(\hat{\xi})$ that flows through dA is

$$dA d\nu dt \int_{\Omega(\hat{\xi}')} I_\nu(\vec{r},\hat{\xi},t)\hat{n}\cdot\hat{\xi} d\Omega(\hat{\xi}')$$

and the quantity

$$F(\hat{\xi}) = \int_{\Omega(\hat{\xi})} I_\nu(\vec{r},\hat{\xi},t)\hat{n}\cdot\hat{\xi} d\Omega(\hat{\xi})$$

is referred to as the spectral flux and has units of $Wm^{-2}.f^{-1}$. Consider the specific form of this flux quantity defined for horizontal surfaces, i.e. $\hat{n}\cdot\hat{\xi} = \cos\theta$, to illustrate the idea. In this case $\hat{n}\cdot\hat{\xi} = \cos\theta$. For an integral performed over a full hemisphere, $Omega(\hat{k}) = 2\pi$, then

$$F = \int_0^{2\pi} d\phi \int_0^1 I(\mu,\phi)\mu d\mu. \tag{1.7}$$

is the hemispheric flux (defined with respect to the horizontal surface whose normal is $\hat{k}$). The net flux is

$$F = \int_0^{2\pi} d\phi \int_{-1}^1 I(\mu,\phi)\mu d\mu. \tag{1.8}$$

Add other moments Other moments - mean intensity, provide some simple integral examples, azimuthal symmetric; isotropic, etc ...

Example 3: The intensity and flux from the sun

The sun radiates approximately as a blackbody. This radiation is isotropic and we set this as a value of $I_\odot$. However, at Earth, this is confined to a very small solid angle $\Omega_\odot$. The flux from the sun on a surface perpendicular to the flow (say this surface is aligned along $\hat{k}$) is this

$$F_\odot = I_\odot \Omega_\odot$$

1.6 Selected Theorems

Theorem 1: The radiance Invariance Law

Theorem 1: The m^2 law for radiance

Fig. 1.4 The geometric setting for the n^2 law.

Consider the situation shown in Fig. 1.4 flowing onto a surface defined by a discontinuity in refractive index m. At the surface

$$\frac{P_1}{A_1} = F_1 = F_2 = \frac{P_2}{A_2}$$

Schnell's law predicts that

$$\begin{aligned} m_1 \sin\theta_1 &= m_2 \sin\theta_2 \qquad (\theta_1, \theta_2 \quad \text{small} \quad \text{by} \quad \text{hypothesis}) \\ m_1\theta_1 &= m_2\theta_2 \end{aligned}$$

and it follows that

$$m_1^2 \Omega_1 = m_2^2 \Omega_2$$

where we make use of our small cap approximation $\Omega = \pi\theta^2$. Since

$$\Omega_1 I_1 = F_1 = F_2 = I_2 \Omega_2$$

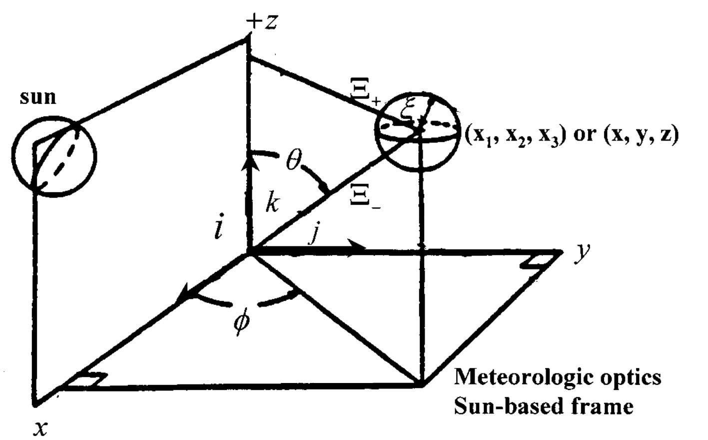
we obtain

$$\frac{I_1}{m_1^2} = \frac{I_2}{m_2^2}$$

Thus we take I/m^2 as the intensity when we are interested in propagation through a medium in which m is varying. The radiance from one m environment (like the atmosphere) to another m environment (like the ocean) thus needs to be adjusted by this m^2 factor.

Nuclear Reactors	Determination of neutron distributions in reactor cores Shielding against intense neutron and gamma radiation
Astrophysics	Diffusion of light through stellar atmospheres (radiative transfer) Penetration of light through planetary atmospheres
Rarefied gas dynamics	Upper atmosphere physics Sound propagation Diffusion of molecules in gases
Charged particle transport	Multiple scattering of electrons Gas discharge physics Diffusion of holes and electrons in semiconductors Development of cosmic ray showers
Transport of electromagnetic radiation	Multiple scattering of radar waves in a turbulent atmosphere Penetration of X-rays through matter
Plasma physics	Microscopic plasma dynamics Microinstabilities Plasma kinetic theory
Other	Traffic flow (transport of vehicles along highways) Molecular orientations of macromolecules The random walk of undergraduates during registration

Table 1.1: Applications of Transport Theory



+z
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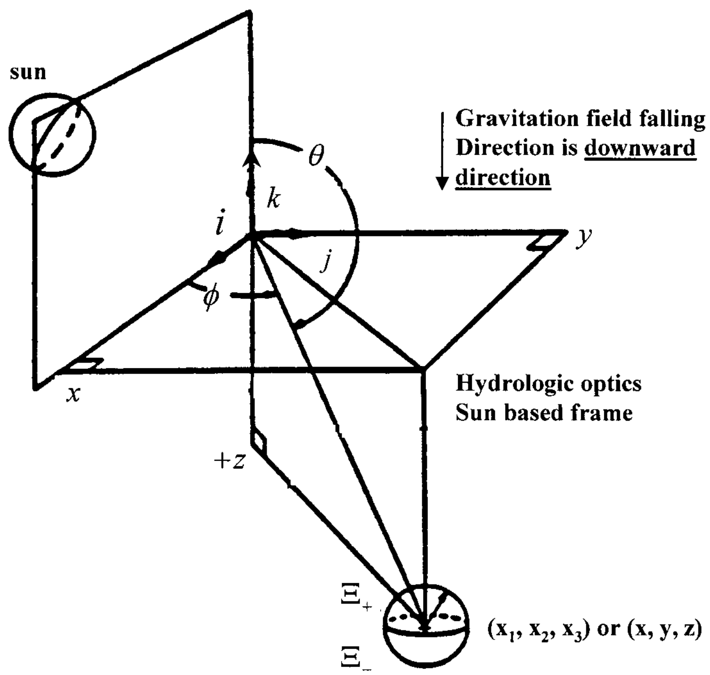
(x1, x2, x3) or (x, y, z)
Meteorologic optics
Sun-based frame



sun
Gravitation field falling
Direction is downward
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Hydrologic optics
Sun based frame
+z
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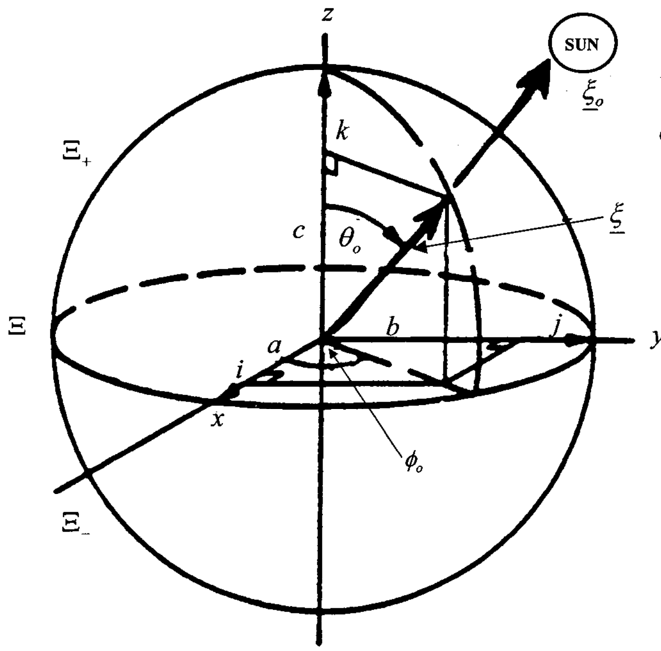
(x1, x2, x3) or (x, y, z)



Unit sphere and directional vector $\vec{\xi}$

Define

$\vec{\xi} = (a, b, c)$

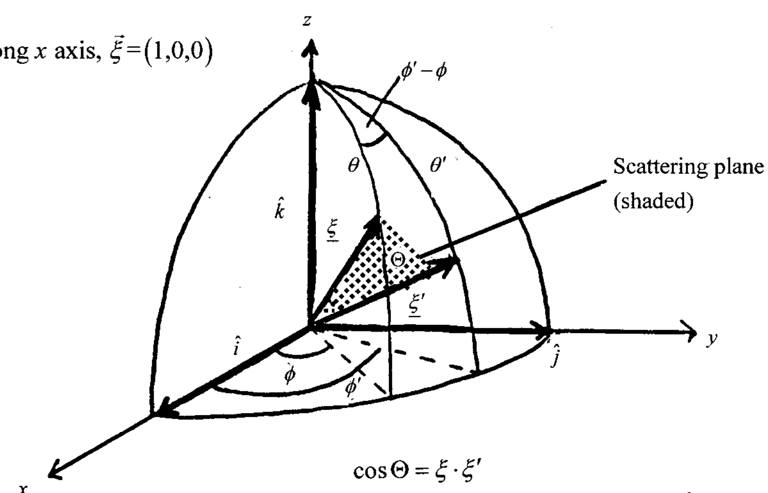
$a = \hat{i} \cdot \vec{\xi} = \sin\theta \cos\phi$

$b = \hat{j} \cdot \vec{\xi} = \sin\theta \sin\phi$

$c = \hat{k} \cdot \vec{\xi} = \cos\theta$

Use subscript "o" to denote direction of the sun

Example $\vec{\xi}$ along x axis, $\vec{\xi} = (1,0,0)$

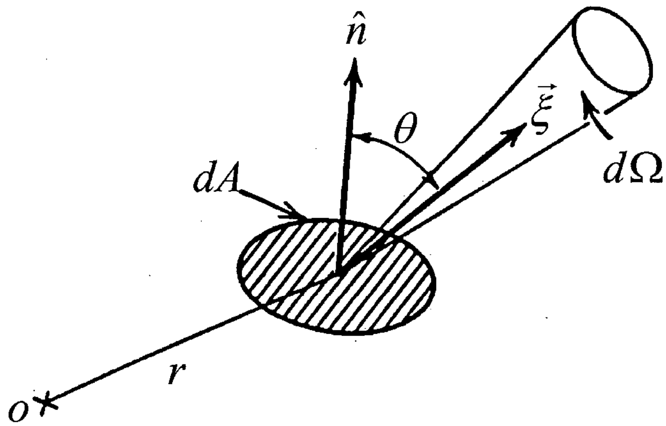


$\cos\Theta = \underline{\xi} \cdot \underline{\xi}'$

$\underline{\xi} = \hat{i}\sin\Theta\cos\phi + \hat{j}\sin\Theta\sin\phi + \hat{k}\cos\Theta$

Then with some manipulation:

$$\underline{\xi} \cdot \underline{\xi}' = \mu\mu' + \left(1-\mu^2\right)^{\frac{1}{2}}\left(1-\mu'^2\right)^{\frac{1}{2}}\cos\left(\phi'-\phi\right) = \left(\mu, \phi, \mu', \phi'\right)$$


$\hat{n}$
θ
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o