A little geometry

Frame of reference is required:

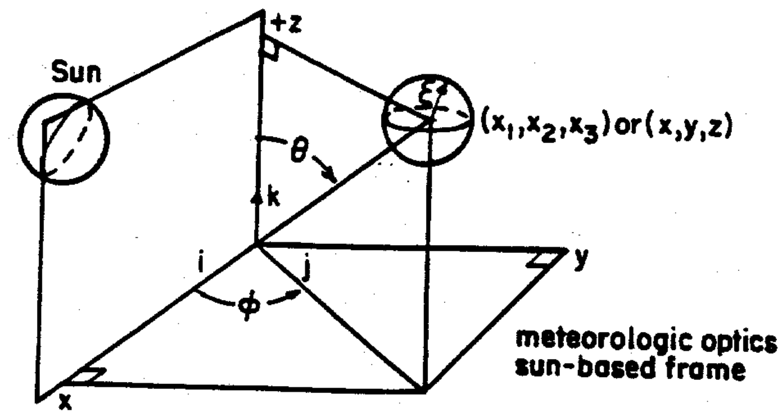
We assume a cartesian coordinate system and orientate this such that the x-axis points to the sun

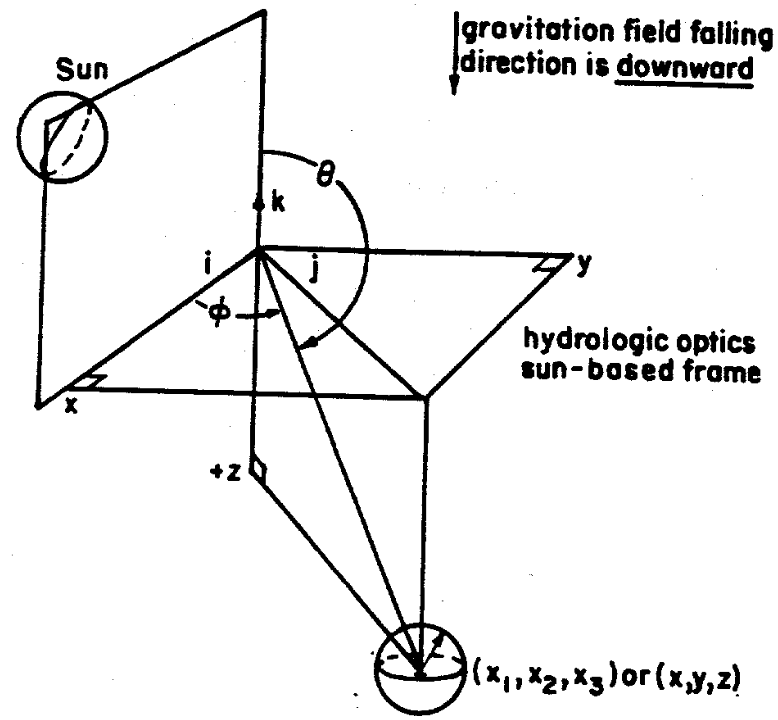
We will encounter two forms of this 'sun-based' frame of reference - one that assumes the z-axis to be altitude (I.e increasing upwards) and another with z increasing downwards (and thus a measure of depth).

We can specify any position in this space as given by a general position vector $\vec{\mathbf{r}}$ by the coordinates of its tip (x,y,z)

The unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are the unit vectors defined along the x,y, and z axes

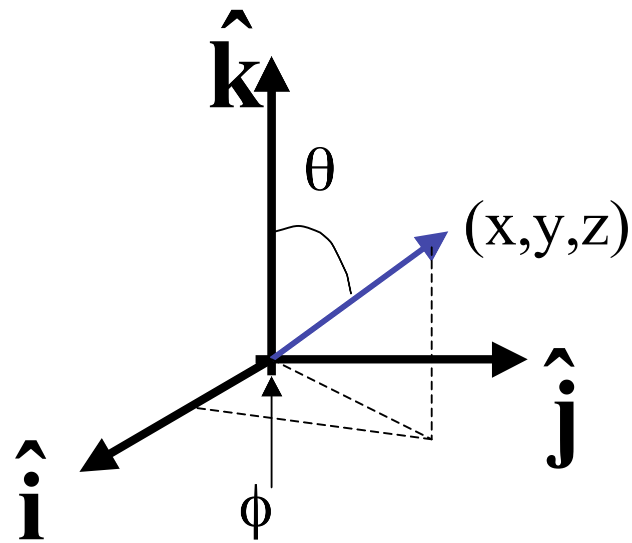
Two angles are also elementary to our considerations - the zenith angle θ and azimuth angle ϕ (measured from the x axis)





Direction

Consider a hypothetical unit sphere about some origin. We define the direction vector as that unit vector that extends from the origin to some point (x,y,z) on that sphere.



$$\vec{\xi} = \frac{\vec{r}}{|r|} = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{1/2}}$$

For a vector of unit length ($|r| = 1$):

$$x = \vec{r} \bullet \hat{i} = \cos\phi \sin\theta$$

$$y = \vec{r} \bullet \hat{j} = \sin\phi \sin\theta$$

$$z = \vec{r} \bullet \overleftarrow{k} = \cos\theta = \mu$$

So that we can express a unit direction vector

$$\vec{\xi} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$$

Important application - angle formed between two directions

$$\cos\Theta = \vec{\xi} \bullet \vec{\xi}'$$

$$\cos\Theta = \mu\mu' +$$

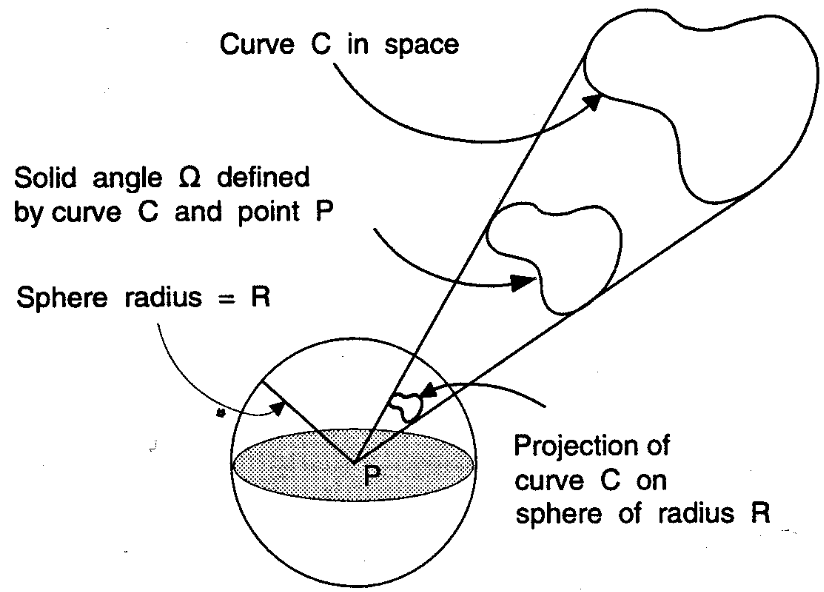
$$(1 - \mu^2)^{1/2}(1 - \mu'^2)^{1/2} \cos(\phi' - \phi)$$

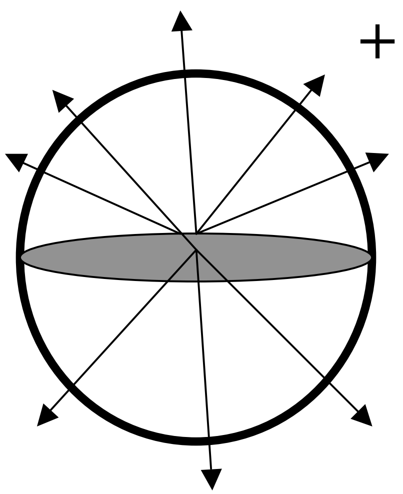
Solid Angle

Consider a unit sphere with some opening of area dA as shown:

solid angle subtended by the opening at origin = area of the opening on the unit sphere

Small element of area is
$da \times db = \sin\theta d\theta \times d\phi = d\Omega$





Establish this unit sphere relative to some direction, such as local vertical - establishes two hemispheres (upper and lower)

Solid Angle Examples

Example 2.2: Solid Angle

- The solid angle of a spherical cap defined by the angle θ is

$$\Omega(D) = \int_0^{\theta} \sin\theta d\theta \int_0^{2\pi} d\phi$$

$$= 2\pi[1 - \cos\theta]$$

For small θ, $\cos\theta \rightarrow 1 - \theta^2/2 + \ldots$ and

$$\Omega(D) = \pi\theta^2$$

For $\theta = -\pi$, the solid angle of a sphere is

$$\Omega(D) = 4\pi.$$

- The solid angle of a spherical segment is

$$\Omega(D) = \int_{\theta_1}^{\theta_2} \sin\theta d\theta \int_0^{2\pi} d\phi$$

$$= 2\pi[\cos\theta_1 - \cos\theta_2]$$

- The solid angle of the sun is

$$\Omega_{\odot} = \pi\theta^2$$

where as we shall see later, $\theta \approx r_{\odot}/R_{S_E}$, and

$$\Omega_{\odot} = \pi\left(\frac{0.7 \times 10^6}{1.5 \times 10^8}\right)^2 \approx 0.684 \times 10^{-4} \quad \text{steradian}$$

Radiant flux - a basic quantity*

Radiant flux along $\vec{\mathbf{r}}$

Photon speed

$$\mathbf{P(\vec{r}) = h\nu \times n(\vec{r}) \times c \times dA} \qquad W(\mu m)^{-1}$$

Photon energy

Photon # density

Area element orthogonal to $\vec{\mathbf{r}}$

F=P/dA=n(r)c hν

n(r)c=F/hν

For F=0.1 Wm^{-2} $(\mu m)^{-1}$, λ=0.5 µm, c=3X10^8 ms^{-1}, h=6.6X10^{-34}Js/photon, then

n(r)c = 0.1 x 0.5 x 10^{-6} /(6.6x10^{-34} x 3x10^{-8})=2.5 x 10^{17}

photons of λ=0.5 µm flow per second through a unit area producing 0.1 Watt of power per $(\mu m)^{-1}$

* but one not directly measurable

Basic measurement concepts -radiance

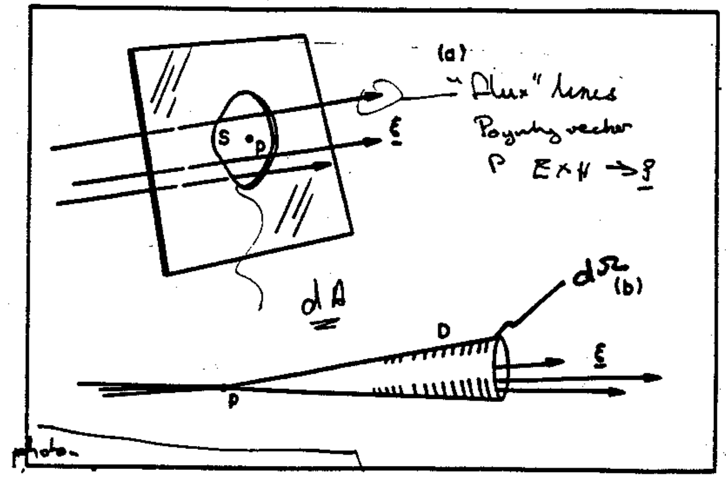
Two quantities thus follow from $P(\vec{\mathbf{r}})$:

Area density radiant flux:

$$F_A = P(\vec{\mathbf{r}})/dA\,(\vec{\boldsymbol{\xi}}) \qquad Wm^{-2}\,(\mu m)^{-1}$$

Solid angle density

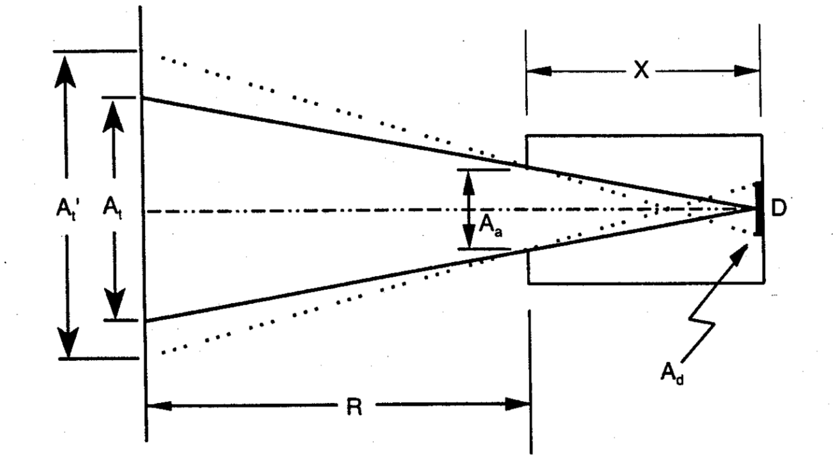
$$F_\Omega = P(\vec{\mathbf{r}})\,/\,d\Omega\,(\vec{\boldsymbol{\xi}}) \qquad Wster^{-1}\,(\mu m)^{-1}$$



Basic quantity measured is the **radiance**
$I = P/T$ $\qquad Wm^{-2}ster^{-1}\,(\mu m)^{-1}$
where $T = dA \times d\Omega$
is the instrument throughput.

Radiance or intensity is fundamental since we can measure it and all other relevant parameters of interest to us derive from it.

Basic measurement concepts -radiometer



<u>Key point: radiance is a 'field' quantity being independent of the distance between the instrument and source assuming the FOV is uniformly filled.</u>

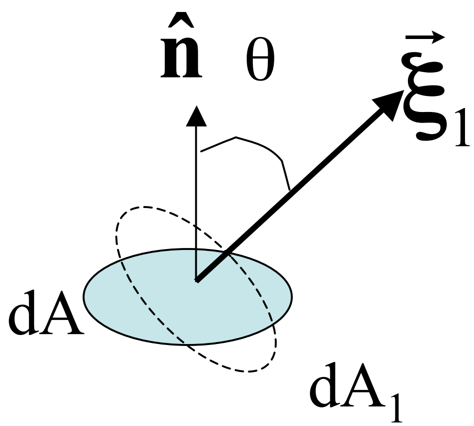
Problem: a simple radiometer is pointed at a wall as shown.
The radiant flux received by a detector D of area A_d at the base of a black tube of length X and aperture area A_a as shown is P. Assuming $A_d \ll X^2$ and $A_a \ll R^2$ what is the averaged radiance of the wall?

Solution: The solid angle subtended by the entrance aperture at the center of the detector is A_a/X^2

The radiance is $P/(A_d\, A_a/X^2)$

Note: (i) the area of the field of view (target area) is A_t at the point p. The total area as seen by the whole detector is A_t' which is larger than A_t. (ii) The radiance is independent of R.

From radiance to irradiance (flux)



- The radiance along direction $\vec{\xi}_1$ is

$$\mathbf{I}_1 = \frac{\mathbf{P}_1}{\mathbf{dA}_1\mathbf{d\Omega}}$$

- The projection of dA_1 onto the surface perpendicular to $\hat{\mathbf{n}}$ is

$$\mathbf{dA} = \mathbf{dA}_1\hat{\mathbf{n}} \bullet \vec{\xi}_1$$

cosθ

- The total energy through dA per unit area

$$\mathbf{F} = \frac{\mathbf{P}_1}{\mathbf{dA}} = \mathbf{I}_1\hat{\mathbf{n}} \bullet \vec{\xi}_1 \mathbf{d\Omega}(\vec{\xi}_1) \qquad \mathrm{Wm^{-2}\ (\mu m)^{-1}}$$

We call F the flux or irradiance

For n sources of radiance I_j, j=1…n along the n directions $\vec{\xi}_j$, j=1…n illuminating the surface, the total rate of energy flow per unit area through the surface dA is just the superposition of each individual source

From radiance to irradiance (flux)

For n sources

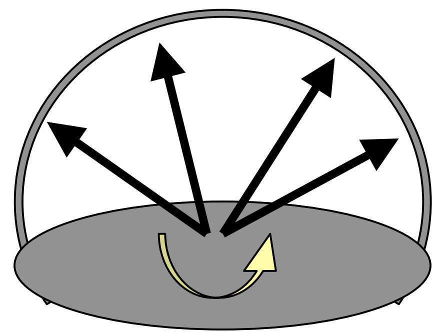
$$I_1 = I(\vec{\xi}_1)$$

$$F(\hat{n}) = \frac{P(\hat{n})}{dA} = I_1 \hat{n} \bullet \vec{\xi}_1 d\Omega(\vec{\xi}_1) + I_2 \hat{n} \bullet \vec{\xi}_2 d\Omega(\vec{\xi}_2) + \ldots$$

$$F(\hat{n}) = \int I(\vec{\xi}')\hat{n} \bullet \vec{\xi}' d\Omega(\vec{\xi}')$$

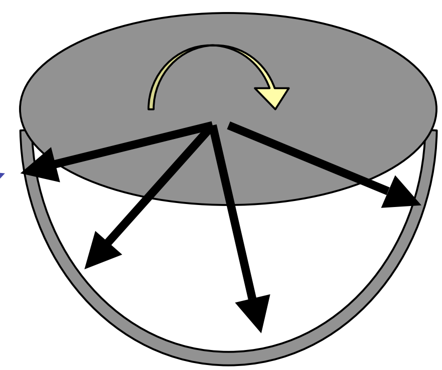
Consider now the special case with $\hat{n} = \hat{k}$

$$d\Omega = \sin\theta d\theta d\phi$$



$$F^{+} = \int_0^{2\pi} d\phi \int_0^{\pi/2} I(\theta,\phi)\cos\theta \sin\theta d\theta$$

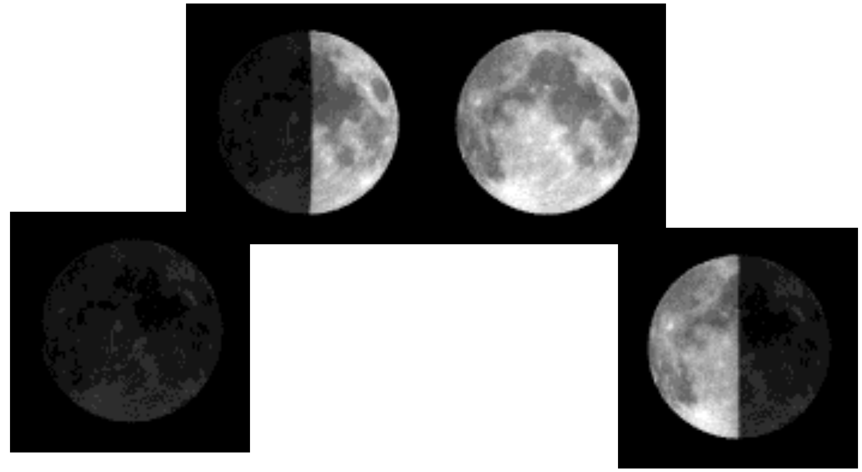
$$(\hat{k} \bullet \vec{\xi} = \cos\theta)$$



$$F^{-} = \int_0^{2\pi} d\phi \int_{-\pi/2}^{0} I(\theta,\phi)\cos\theta \sin\theta d\theta$$

Scattering phase function

Consider an experiment that places a detector at some distance from a scattering volume. We seek to measure the scattered intensity at all such points located around an imaginary sphere that surrounds the scatterer



As before:

$$E_{inc} = E_0 e^{-ikz+i\omega t}$$

$$E_{sca} = S(\Theta)\frac{e^{-ikr+i\omega t}}{kr}$$

$$E_{sca} = S(\Theta)\frac{e^{-ikr+ikz}}{kr}E_0$$

$$I_{sca} = \frac{|S(\Theta)|^2}{k^2 r^2} I_0$$

power received by detector

$$dW = I_{sca}dA = I_{sca}r^2 d\Omega$$

$$= \frac{|S(\Theta)|^2}{k^2} I_0 d\Omega$$

total power scattered

$$W = \int_{\Xi} \frac{|S(\Theta)|^2}{k^2} I_0 d\Omega = I_0 C_{sca}$$

thus

$$C_{sca} = \int_{\Xi} \frac{|S(\Theta)|^2}{k^2} d\Omega$$

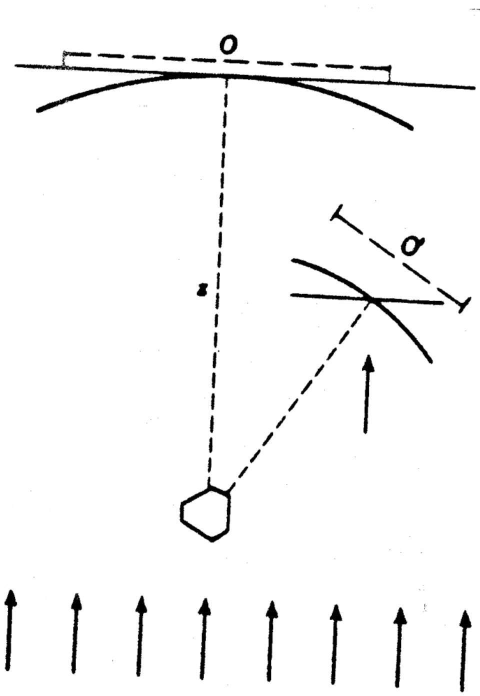
define

$$\frac{1}{4\pi}P(\Theta) = \frac{|S(\Theta)|^2}{C_{sca}k^2}$$

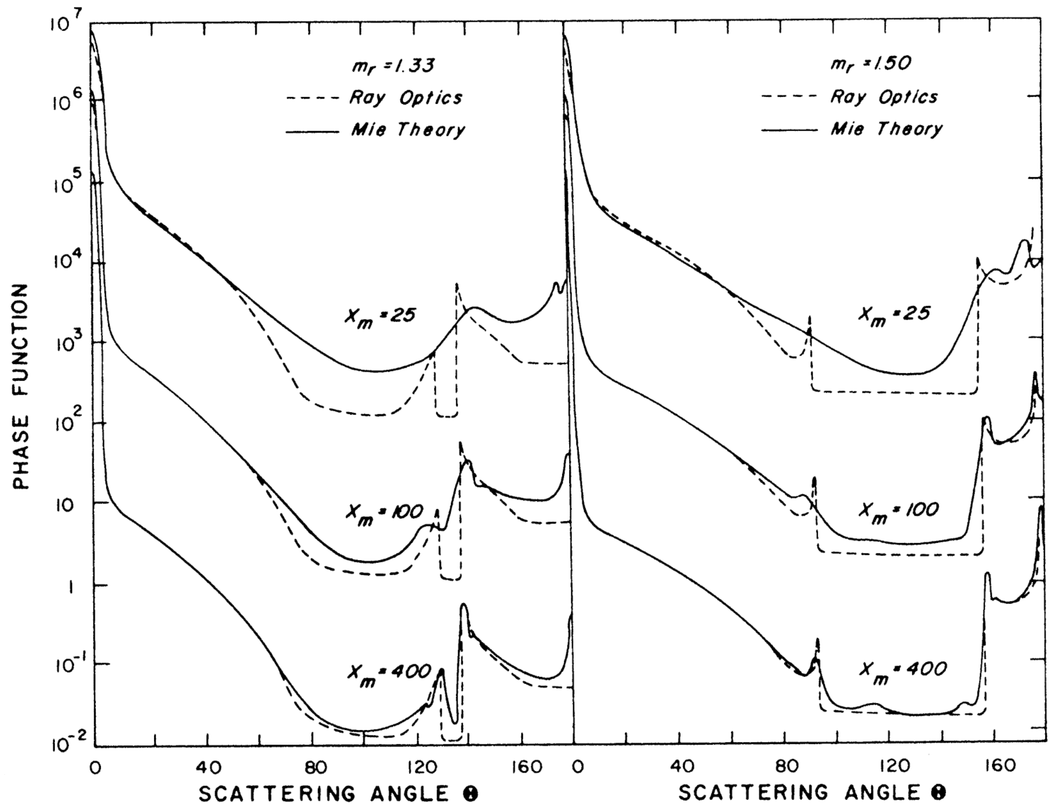
by definition

$$\frac{1}{4\pi}\int_{\Xi} P(\Theta)d\Omega = 1$$

$$I_{sca}(\Theta) = \frac{C_{sca}k^2}{4\pi}P(\Theta)I_0$$



Examples of phase function



http://members.tripod.com/~regenbogen/indexe.htm

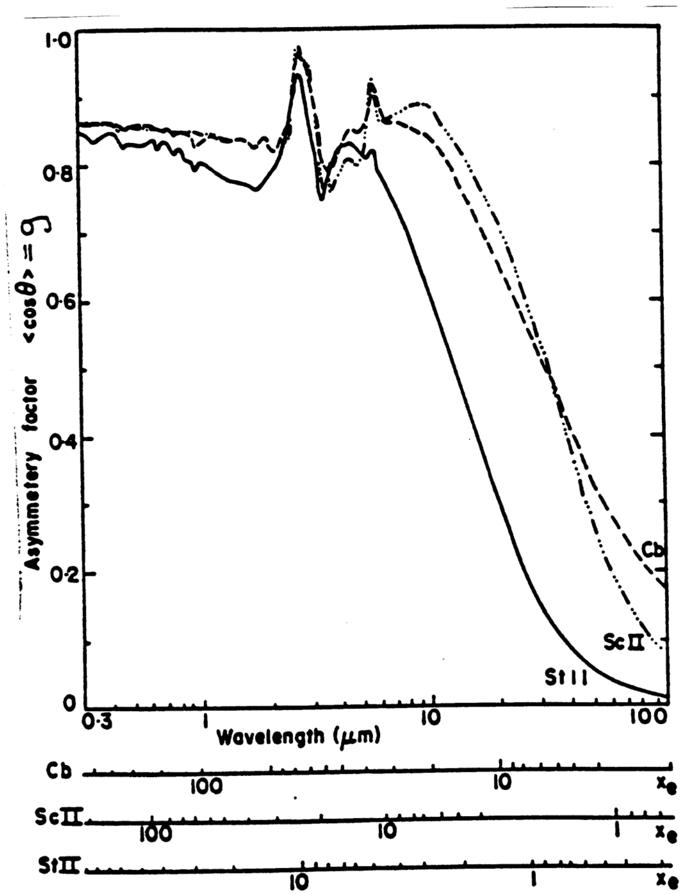
Points to note: the extent of forward scattering & how it increase with x

Optical phenomena like rainbow and glory and where they appear

Smoothing of scattering function for polydispersions

Anticipate the effects of particle absorption

Properties of the phase function



$$g = \frac{1}{2}\int_{-1}^{+1} P(\cos\Theta)\cos\Theta \, d\cos\Theta$$

asymmetry parameter

g=1 pure forward scatter
g=0 isotropic or symmetric (e.g
Rayleigh)
g=-1 pure backscatter

Other properties of the phase function

Define forward scatter & backscatter as

$$f = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^1 P(\theta,\phi)\, d\cos\theta = \frac{1}{2}P(0)$$

$$b = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{-1}^0 P(\theta,\phi)\, d\cos\theta = \frac{1}{2}P(180)$$

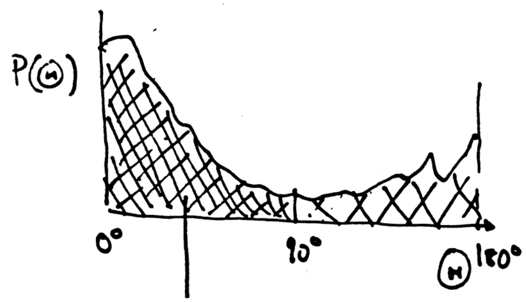
$f + b = 1$ (normalization condition)

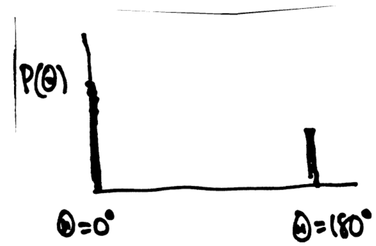
For a simple delta phase function

$$g = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{-1}^1 P(\theta,\phi) \cos\theta\, d\cos\theta = f - b$$

$$b = \frac{1}{2}(1-g)$$

$$f = \frac{1}{2}(1+g)$$





Other (mathematical) forms of phase function

$$P(\cos\Theta) = \sum_{\ell=0}^{N} \chi_\ell P_\ell(\cos\Theta)$$

a particular form of function has

$$\chi_\ell = \frac{(2\ell+1)}{2} g^\ell$$

and with $N = 1$

$$P(\cos\Theta) = (1 + \frac{3}{2} g\cos\Theta)$$

Legendre polynomial expansion

Legendre Polynomial of degree ℓ