Phase Functions

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1 The Phase Function

This is just a short note to describe my understanding of phase functions. It is taken entirely from Liou (2002). In general, the scattering of radiation off a particle is described by the phase matrix $\mathbf{P}(\Theta)$, which is a 4x4 matrix. In the most general case of non-randomly oriented nonspherical particles, $\mathbf{P}(\Theta)$ has 16 independent elements. Each element P_{ij} is a function of the scattering angle Θ, where

$$\cos\Theta \;=\; \mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos\phi - \phi' \tag{1}$$

Thus, $P_{ij}(\mu,\phi,\mu',\phi')$ represents the fraction of scattered radiation of Stokes parameter i, incident from direction (μ',ϕ') upon a scatterer, and scattered into direction (μ,ϕ) with Stokes parameter j. The four Stokes parameters are of course (I,Q,U,V). For a review of polarization see Section 2 of this document.

For unpolarized radiation, we only consider the $P_{11}(\Theta)$ element of $\mathbf{P}(\Theta)$; let us call this simply $P(\Theta)$ and refer to it simply as the *phase function*. However, it would be more accurate to call it the "intensity phase function". The (intensity) phase function is conventionally normalized as follows

$$\int_0^{2\pi}\int_0^{\pi} P(\Theta)\sin\Theta d\Theta d\phi = 4\pi \tag{2}$$

which is equivalent to

$$\int_{-1}^{1} P(\cos\Theta) d\cos\Theta = 2 \tag{3}$$

where oftentimes we write $\mu = \cos\Theta$ for brevity. But note that this is a different μ than in Equation (1)!! Some researchers prefer to express each element $P(\Theta)$ of the phase function as an expansion in Legendre polynomials (spherical harmonics). Then we may write:

$$P(\Theta) \;=\; \sum_{\ell=0}^{N} \chi_\ell P_\ell(\cos\Theta) \tag{4}$$

where P_ℓ are the Legendre Polynomials. Formally $N = \infty$ but in reality the χ_ℓ coefficients often die away after hundreds, thousands, or perhaps tens of thousands of coefficients. Therefore one can typically set a reasonable upper limit N which will retain all the same features of the full phase function. For the full phase function without azimuthal symmetry, one can write:

$$P(\mu, \phi, \mu', \phi') = \sum_{m=0}^{N} \sum_{\ell=m}^{N} \chi_\ell^m P_\ell^m(\mu) P_\ell^m(\mu') \cos m(\phi - \phi') \tag{5}$$

where we define $\chi_\ell \equiv \chi_\ell^0$, and the $P_\ell^m(\mu)$ are the Associated Legendre Polynomials. Also note that $P_\ell = P_\ell^0$. Equation 5 can be derived using the addition theorem of spherical harmonics (see Appendix E of Liou). I should note that the ability to write a phase function as a function of simply Θ is only good for either spherical or randomly oriented particles. For oriented nonspherical particles, the math is much harder, and is not considered further here.

In order to calculate the expansion coefficients for a real phase function, one can use the property that the Legendre polynomials are *orthogonal* on the range $\{-1, 1\}$. Specifically,

$$\int_{-1}^{1} P_i(\mu) P_j(\mu) d\mu = \frac{2}{2\ell + 1} \delta_{i,j} \tag{6}$$

where $\delta_{i,j}$ is the kronecker-delta function; it equals 1 for $i = j$ and 0 for $i \neq j$. One can then show that χ_ℓ is given by:

$$\chi_\ell = \frac{2\ell + 1}{2} \int_{-1}^{1} P_\ell(\mu) P(\mu) d\mu \tag{7}$$

For real phase functions, one must typically compute this integral numerically. [1] To calculate (regular) Legendre polynomials, it is often convenient to use the recursion relation called "Bonnet's recursion formula":

$$P_{\ell+1}(\mu) = \frac{1}{\ell + 1} \left((2\ell + 1) \mu P_\ell(\mu) - \ell P_{\ell-1}(\mu) \right) \tag{8}$$

Then one simply uses the fact that $P_0 = 1$ and $P_1(\mu) = \mu$ and one can obtain all higher $P_\ell(\mu)$ from there.

For many applications, one may consider the azimuthally averaged phase function value for incoming zenith direction μ' and outgoing direction μ. We will denote this as $\tilde{P}(\mu, \mu')$. Note that $P(\Theta)$ is only a function of $\phi - \phi'$. Therefore, the azimuthally averaged intensity phase function is given by:

$$\begin{aligned} \tilde{P}(\mu, \mu') &= \frac{1}{2\pi} \int_0^{2\pi} P(\mu, \phi, \mu', \phi') d(\phi - \phi') \\ &= \sum_{\ell=0}^{N} \chi_\ell P_\ell(\mu) P_\ell(\mu') \end{aligned} \tag{9}$$

[1] One can use Simpson's rule or something similar, but *Gaussian Quadrature* also works well.

The latter equality is shown easily from Equation (5).

Note that, often times we refer to the *asymmetry factor*, g, of a phase function. For an azimuthally-averaged phase function, this is defined as the first moment of the phase function in $\cos\Theta$ (i.e., the mean of $\cos\Theta$ when treating the phase function as a probability distribution function):

$$g \equiv \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\cos\Theta \; d\cos\Theta \tag{10}$$

It is simple to show that $g = \chi_1/3$.

Finally, a convenient one-parameter phase function called the Henyey-Greenstein phase function is sometimes used. It is given by

$$P_{HG}(\Theta, g) = \frac{1-g^2}{(1+g^2-2g\cos\Theta)^{3/2}} \tag{11}$$

This form has the logical feature that the asymmetry parameter is truly given by the parameter g. It also has a very simple Legendre expansion:

$$P_{HG}(\Theta, g) = \sum_{\ell=0}^{\infty}(2\ell+1)g^{\ell}P_{\ell}(\cos\Theta) \tag{12}$$

2 Review of Polarization Description

Let us briefly review the mathematical description of polarization. An electromagnetic wave can in general be written in terms of its electric field as

$$\vec{E} = E_x\hat{x} + E_y\hat{y} \tag{13}$$

where

$$\begin{aligned} E_x &= E_{x_0}e^{i(kz-\omega t+\phi_x)} \\ E_y &= E_{y_0}e^{i(kz-\omega t+\phi_y)} \quad . \end{aligned}$$

It is implicit that one takes the real part of $\vec{E}$ to obtain the physical field. We can equally well describe this radiation by four *scalar* quantities, the *Stokes parameters* which are defined as follows:

$$I = \langle E_{x_0}^2 + E_{y_0}^2 \rangle \tag{14a}$$
$$Q = \langle E_{x_0}^2 - E_{y_0}^2 \rangle \tag{14b}$$
$$U = 2\langle E_{x_0}E_{y_0}\cos(\phi_x - \phi_y)\rangle \tag{14c}$$
$$V = 2\langle E_{x_0}E_{y_0}\sin(\phi_x - \phi_y)\rangle, \tag{14d}$$

where $\langle \ldots \rangle$ denotes a time average. Q and U both represent linear polarization, while V represents circular polarization. For quasimonochromatic light, each component of Equation (14) is understood to be averaged over the entire frequency band.

The Stokes parameters I and V are unchanged under rotations of the $\hat{x} - \hat{y}$ plane, but Q and U are not. If we rotate the $\hat{x} - \hat{y}$ axes through an angle θ, the Stokes parameters change as

$$\begin{aligned} Q' &= Q \cos 2\theta + U \sin 2\theta \\ U' &= -Q \sin 2\theta + U \cos 2\theta \end{aligned} \tag{15}$$

The angle $\alpha \equiv \frac{1}{2} \arctan \frac{U}{Q}$ transforms to $\alpha - \theta$ under the rotation; hence it defines a constant direction in space, which is interpreted as the axis of polarization. Finally, the fraction of polarization is typically denoted by

$$\Pi = \frac{\sqrt{Q^2 + U^2 + V^2}}{I} . \tag{16}$$

For a fully polarized signal, $\Pi = 1$. A partially polarized signal has $0 < \Pi < 1$. An unpolarized signal has $\Pi = Q = U = V = 0$.

References

Liou, K. N., 2002. "An Introduction to Atmospheric Radiation". Academic Press c/o Elsevier Science, San Diego, CA.