AT622 Section 15
Radiative Transfer Revisited: Two-Stream Models

The goal of this section is to introduce some elementary concepts of radiative transfer that accounts for scattering, absorption and emission and introduce simple ways of solving multiple scattering problems. We introduce simple models to solve the relevant radiative transfer equation and demonstrate how they offer a glimpse at the intricate way in which the radiance field depends on the properties of the scattering and absorbing medium.

15.1 Scattering as a Source of Radiation

Photons flowing along a given direction are removed by single scattering as Beer's law predicts. However, these photons can actually reappear again along that same direction when scattered a multiple number of times. In fact, many of the scattering media of interest to studies of the atmosphere are multiple scattering media, that is media containing a sufficient number of scatterers that photons traversing it are likely to be scattered more than once. Multiple scattering of sunlight, for instance, gives rise to many observable phenomena that cannot be explained from single scattering arguments alone. For example, single scattering predicts a sky that is of uniform brightness and color contrary to what we observe. The whiteness and brightness of clouds is also a result of multiple scattering. Reflection of visible and microwave radiation from various surfaces is largely influenced by multiple scattering. Multiple scattering is thus relevant to many topics and we need to develop a mathematical description of how these photons reappear along the reference direction in order to account for it.

Consider a beam of monochromatic radiation flowing along a direction defined by the vector $\vec{\xi}'$ illuminating a small volume located at $\vec{r}$ and of length *ds* containing scattering particles (Fig. 15.1). The volume is taken to be small enough that only single scattered photons emerge from it. The incremental

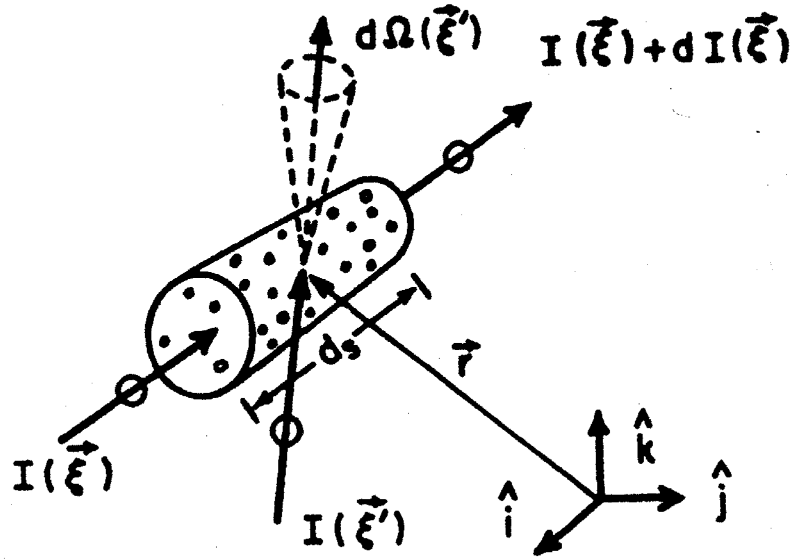

Fig. 15.1 Geometry for scattering of diffuse light. $\vec{\xi}'$ is the unit vector that defines the direction of the flow and $\vec{r}$ is the vector that specifies the position of the volume element relative to an origin point.

increase in intensity along the direction specified by $\vec{\xi}$ due to the scattering of this incident beam is, by virtue of the definition of the phase function[1] given in Section 13,

$$\delta I(\vec{r},\vec{\xi}) = \sigma_{sca} ds \frac{P(\vec{r},\vec{\xi},\vec{\xi}')}{4\pi} I(\vec{r},\vec{\xi}') d\Omega(\vec{\xi}'), \tag{15.1}$$

where σ_{sca} is the volume scattering coefficient given earlier and the wavelength dependence on all quantities is understood. The total contribution to $I(\vec{r},\vec{\xi})$ by scattering of the complete diffuse field surrounding the volume is given by the integral of Eqn. (15.1), namely

$$dI(\vec{r},\vec{\xi}) = \sigma_{sca} ds \int_{4\pi} \frac{P(\vec{r},\vec{\xi},\vec{\xi}')}{4\pi} I(\vec{r},\vec{\xi}') d\Omega(\vec{\xi}'), \tag{15.2}$$

which leads to the following definition

$$J(\vec{r},\vec{\xi}) = \tilde{\omega}_o \int_{4\pi} \frac{P(\vec{r},\vec{\xi},\vec{\xi}')}{4\pi} I(\vec{r},\vec{\xi}') d\Omega(\vec{\xi}')$$

such that

$$dI(\vec{r},\vec{\xi}) = \sigma_{sca} ds\, J(\vec{r},\vec{\xi}). \tag{15.4}$$

The quantity $J(\vec{r},\vec{\xi})$ is the source of radiation due to scattering of diffuse light (sometimes this source is referred to as *virtual emission)* and $\tilde{\omega}_o$ is the single scatter albedo defined as

$$\tilde{\omega}_o = \frac{\sigma_{sca}}{\sigma_{ext}}$$

and varies between zero for pure absorption and unity for pure scattering (the latter condition is known as conservative scattering) such that the quantity 1 - $\tilde{\omega}_o$ is the fraction of the incident radiation that is absorbed by the small volume element under consideration.

The monochromatic radiative transfer equation defines the net change in intensity of a beam as it traverses the path element *ds* through a small volume. The change in intensity as the beam traverses a volume of atmosphere that both absorbs and scatters {and emits} radiation is

$$dI = dI(extinction) + dI(scattering) + \{dI(emission)\}, \tag{15.5}$$

or

$$\frac{dI(\vec{r},\xi)}{ds} = -\sigma_{ext}[I(\vec{r},\xi) - J(\vec{r},\xi)] + \{dI(emision)\}, \tag{15.6a}$$

[1] The phase function is a bidirectional scattering function that is entirely analogous to the bidirectional functions.

after collecting the extinction term and Eqn. (15.4) for scattering. It is relevant to note the similarity of this equation to Eqn. (4.8a,b) except that the scattering source in Eqn. (15.6a) is replaced by the Planck function in (4.8b).[2]

The radiative transfer equation relevant to a horizontally stratified atmosphere is

$$\mu \frac{dI(z,\theta,\phi)}{dz} = -\sigma_{ext} I(z,\theta,\phi) + \frac{\sigma_{sca}}{4\pi} \int_{4\pi} P(z,\theta,\phi,\theta'\phi') I(r,\theta'\phi') d\Omega' + \sigma_{abs} B(T) . \tag{15.6b}$$

where if it were not for the presence of the integral term, this equation would be a mere differential equation and the theory of multiple scattering would have been worked out and forgotten long ago.

Example 15.1: Virtual Sources?

The problem of solar radiation multiply scattered by cloud or aerosol is more conveniently posed in terms of a source of collimated light that enters the cloud and by scattering creates a virtual source of *diffuse* radiation. This leads to an additional source term in the equation of transfer, which may be derived as follows. Consider Eqn. (15.6b), and suppose that the *global* intensity $I(z, \theta, \phi)$ may be expressed as two components:

$$I(z,\theta,\phi) = I_*(z,\theta,\phi) + I_o(z,\theta,\phi)$$

one for the diffuse field I_* and the second

$$I_0(z,\theta,\phi) = I_\odot(z)\delta(\theta - \theta_\odot)\delta(\phi - \phi_\odot)$$

for a collimated beam of intensity $I_\odot$ along the solar direction $\theta = \theta_\odot$, $\phi = \phi_\odot$. Substitution leads to

$$\mu \frac{d}{dz} I_* + I_o = -\sigma_{ext}(I_* + I_o) + \frac{\sigma_{sca}}{4\pi} \int P(I_* + I_o) d\Omega'$$

or

$$\mu \frac{dI_*}{dz} = -\sigma_{ext} I_* + \frac{\sigma_{sca}}{4\pi} \int P I_* d\Omega + I_\odot P(\theta,\phi,\theta_\odot,\phi_\odot)$$

and

$$\mu_\odot \frac{dI_0}{dz} = -\sigma_{ext} I_\odot, \text{ where } I_\odot(z) = I_\odot(z_r) e^{-\sigma_{ext}(z_r - z)/\mu_\odot}$$

[2] We can also follow the procedure of Section 4.3 to obtain an integral equation of transfer that is analogous to Eqn. (4.10). However, there is a fundamental difference between this and the equivalent integral equation that follows in this way. In Eqn. (4.10) the source function appearing in the integrand is known as *a priori* (assuming that the temperature distribution along the path is known) and the solution requires a straightforward integration of known functions. For scattering, the source function appearing in the integrand unfortunately contains the desired intensity and cannot be evaluated *a priori* unless some approximation is made. The presence of the intensity in the definition of J is what complicates the problem of multiple scattering and why a host of different approaches exist to overcome it.

15.2 Multiple Scattering: A Natural Method of Solution

A natural solution to problems of multiply scattered light in the atmosphere is one that decomposes the light field into components that can be identified with the number of times a photon has been scattered. This is termed the method of *orders of of scattering* and Fig. 15.2 provides the general geometric setting for discussing this approach. Suppose light of intensity I_o enters the medium along the direction $\vec{\xi}$ at the point $\vec{r}_o$. The amount of radiation leaving the distant point $\vec{r}_o$ along $\vec{\xi}$ is

$$I^0(\vec{r},\vec{\xi}) = I_o(\vec{r}_o,\vec{\xi})T(\vec{r}_o,\vec{r},\vec{\xi}) \tag{15.7}$$

where $t(\vec{r}_o,\vec{r},\vec{\xi})$ is the transmission function defined by the path $\vec{r}_o \rightarrow \vec{r}$ along $\vec{\xi}$ as shown in Fig. 15.2. We refer to I^0 as the *reduced* or *unscattered* intensity. When some light $I^o(\vec{r}',\vec{\xi}')$ at an intermediate point $\vec{r}'$ undergoes a scattering event, a first order or primary scattered intensity is generated at that point by an amount defined by Eqn. (15.3). This amount of intensity per unit length of path is

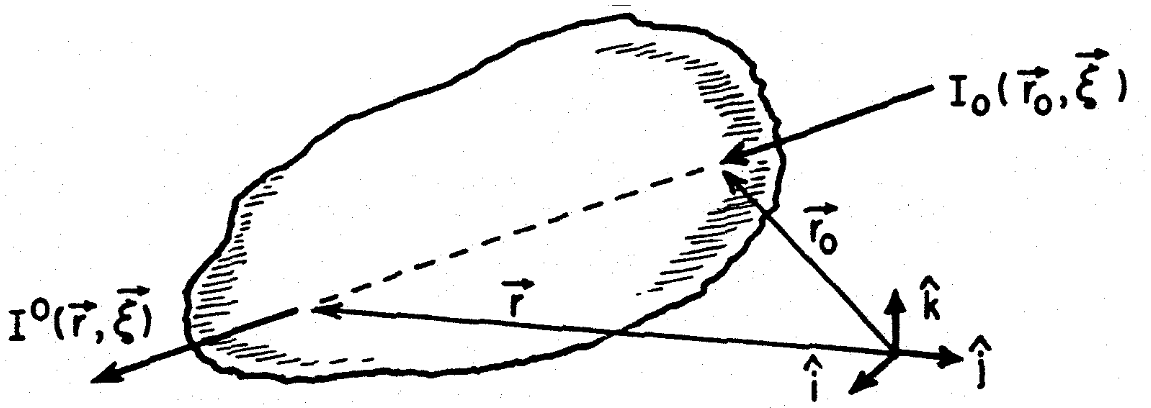

Fig. 15.2 Geometry for orders of scattering and geometry of a plane parallel atmosphere for computing the primary scattered intensity induced by a collimated source of solar radiation of intensity I_o.

$$I_*^1(\vec{r}',\vec{\xi}) = J^1(\vec{r}',\vec{\xi})\sigma_{ext}$$

where, according to Eqn. (15.3), J^1 may be considered as the source associated with the primary scattering of I^0. It therefore follows that the radiation from primary scattering of light from all directions is

$$I_*^1(\vec{r}',\vec{\xi}) = \sigma_{sca}\int_{4\pi}\frac{P(\vec{r}',\vec{\xi},\vec{\xi}')}{4\pi}I^0(\vec{r}',\vec{\xi}')d\Omega(\vec{\xi}')\,. \tag{15.8}$$

The amount of this primary scattered radiation that is accumulated along the path from $\vec{r}_o \rightarrow \vec{r}$ is

$$I^1(\vec{r},\vec{\xi}) = \int_{\vec{r}_o}^{\vec{r}} I_*^1(\vec{r}',\vec{\xi})T(\vec{r}_o,\vec{r}',\vec{\xi})d\vec{r}'\,. \tag{15.9}$$

It is a simple and somewhat intuitive matter to show that construction of the intensities associated with higher order scattering then follows from the repeated application of Eqns. (15.8) and (15.9) such that

$$I_*^{n+1}(\vec{r}',\vec{\xi}) = \sigma_{sca}\int_{4\pi}\frac{P(\vec{r}',\vec{\xi},\vec{\xi}')}{4\pi}I^n(\vec{r}',\vec{\xi}')d\Omega(\vec{\xi}'), \tag{15.10a}$$

$$I^{n+1}(\vec{r},\xi) = \int_{\vec{r}_o}^{\vec{r}} I_*^{n+1}(\vec{r}',\xi)T(\vec{r}_o,\vec{r}',\xi)d\vec{r}' \tag{15.10b}$$

for each integer order $n = 0, 1, \ldots$ of scattering. The total intensity is therefore the sum of all orders of scattering, namely

$$I = I^0 + I^1 + I^2 + \ldots + I^n + \ldots = \sum_n I^n, \tag{15.11a}$$

which is conveniently written as

$$I = I^0 + I^* \tag{15.11b}$$

where I^* in this case is the total diffuse intensity $I^* = \sum_{n=1} I_n$.

An obvious question to ask is how many orders of scattering are required to approximate the diffuse field to some given accuracy? The general answer to this question depends on how many particles there are in the volume and on how efficiently the particles scatter the radiation. A rough idea of the effect of the single scatter albedo on scattering is given by the following arguments. Suppose $\overline{I}^0$ is an upper bound on I^0. Then from Eqn. (15.8)

$$I_*^1(\vec{r}',\vec{\xi}) \le \overline{I}^0\sigma_{sca}\frac{1}{4\pi}\int P(\vec{r}',\vec{\xi},\vec{\xi}')d\Omega(\vec{\xi}') = \overline{I}^0\sigma_{sca} \tag{15.12}$$

by virtue of the phase function renormalization condition Eqn. (15.9) then becomes

$$I^1(\vec{r},\vec{\xi}) = \int I_*^1(\vec{r}',\vec{\xi})e^{-\sigma_{ext}|\vec{r}-\vec{r}'|}d\vec{r}'$$

where the exponential factor is the transmission function for the path of length $d = |\vec{r}-\vec{r}'|$. From the condition on I_*^1 it follows that

$$I^1(\vec{r},\vec{\xi}) \le \overline{I}^0\omega_o(1-e^{-\sigma_{ext}d}) \le \overline{I}^0\tilde{\omega}_o .$$

Repeating this procedure for the next order of scattering leads to

$$I^2(\vec{r},\vec{\xi}) \le \overline{I}^0\tilde{\omega}_o$$

and

$$I^n(\vec{r},\vec{\xi}) \le \overline{I}^0\tilde{\omega}_o^n \tag{15.13}$$

for every scattering order n. With the following notation

$$I^{(k)}(\vec{r},\vec{\xi}) \quad \text{for} \quad \sum_{n=0}^{k} I^{n}(\vec{r},\vec{\xi}),$$

it follows that the difference $\Delta = I(\vec{r},\vec{\xi}) - I^{(k)}(\vec{r},\vec{\xi})$ is

$$\Delta = \sum_{j=k+1} I^{j}(\vec{r},\vec{\xi}) \le \overline{I}^{0} \sum_{j=k+1} \tilde{\omega}_o$$

or

$$\Delta \le \overline{I}_o \tilde{\omega}_o^{j+1} \sum_{j=0} \tilde{\omega}_o^{j} = \overline{I}^{0} \frac{\tilde{\omega}_o^{j+1}}{(1-\tilde{\omega}_o)}. \tag{15.14}$$

Example 15.2: How many times does a photon get scattered?

Consider the example with $\tilde{\omega}_o$ = 0.5 and suppose that we require I^k to differ from the actual intensity by an amount no larger than 1% of $\overline{I}^{0}$. It follows that $\Delta/\overline{I}^{0} \le$ 0.01 and that

$$0.01 \le \frac{0.5^{j+1}}{0.5}$$

or j = 7 for the nearest integer value. Thus only 7 orders of scattering are required to model the diffuse intensity with a 1% accuracy when $\tilde{\omega}_o$ = 0.5. This simple exercise offers a clear illustration of the significance of $\tilde{\omega}_o$ to multiple scattering. We infer that the number of scatterings required to represent the total intensity decreases as the absorption by the particle increases (or as $\tilde{\omega}_o \to 0$). For example, many orders of scattering contribute to the total radiation field in clouds at solar wavelengths where $\tilde{\omega}_o$ > 0.9 but relatively few scatterings contribute at the infrared wavelengths where $\tilde{\omega}_o$ < 0.5

15.3 The Two-Stream Approximation

On examination of the equation of transfer, which includes scattering in either its interodifferential or its integral form, one is confronted with the complicating presence of the integral term that involves an integration over the direction variable. In fact if it weren't for this term, the equation of transfer would be but a mere differential equation and the theory of multiple scattering would have been worked out and forgotten long ago. Thus the essence of the simplification that is introduced by the class of simple models discussed here is to approximate, in some way, the angular shape of the radiance field so as to introduce some approximation to this integral term. To this end there is a property of the radiance field that is utilized to great benefit by these approximate methods although it is not often explicitly realized. This property is illustrated in Figs. 15.3a and b in which the zenith radiance distribution is shown on descent into the sea (Fig. 15.3a) and deep in a thick cloud (Fig. 15.3b). It is apparent that this radiance structure approaches some sort of asymptotic form with increasing depth into the "medium". Eventually some steady distribution is reached and all radiances decrease at the same exponential rate with increasing depth ultimately shrinking down in size but preserving its shape. It is also apparent that this *asymptotic distribution* can be described as some simple function of zenith angle (this is the basis of the diffusion approximations, which we will not discuss here).

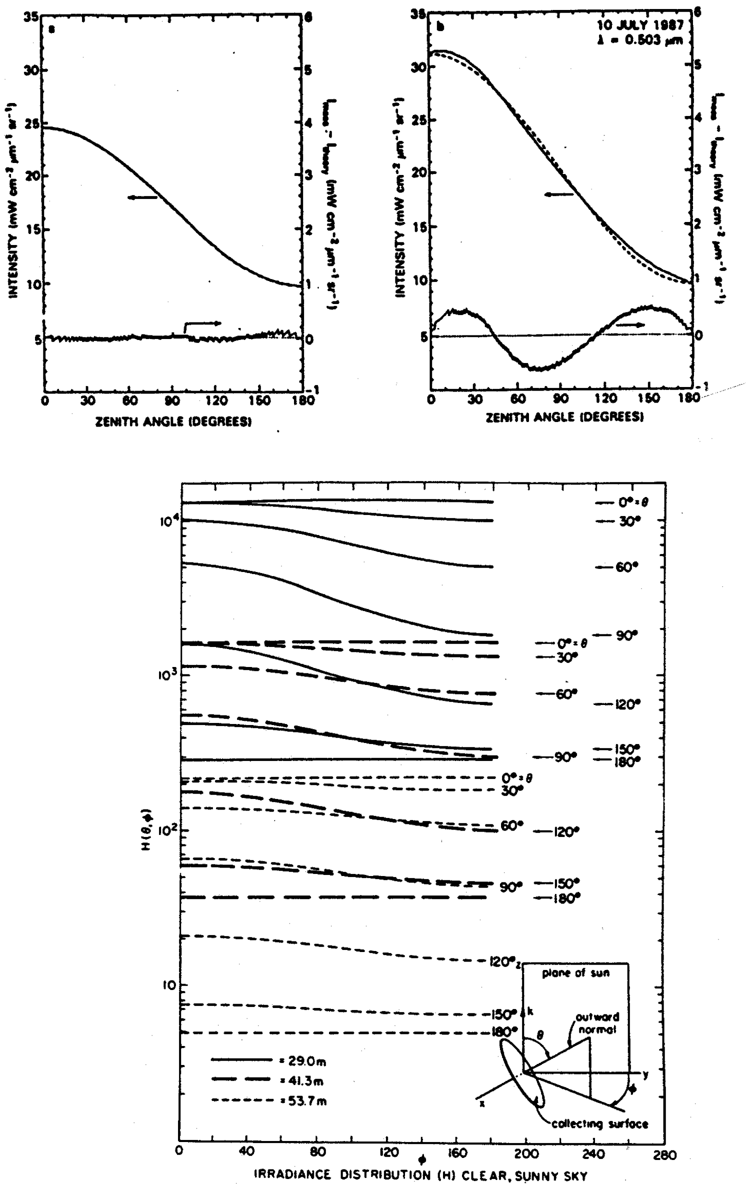

Fig. 15.3 (a) The flux distribution on a clear sunny day at three indicated depths in Lake Pend Orielle, Idaho (adapted from Preisendorfer, 1976). These fluxes are defined for a collecting surface inclined at an angle θ as shown in the inset. (b) Measured intensity as a function of zenith angle obtained from a scanning radiometer on an aircraft as it flew through the center of a deep stratiform cloud. The lower curve is the difference between measurement and a simple cosine of zenith angle variation (King et al., 1990).

While we can approach the development of the two-stream equations in a number of different ways, the end result is always the same, namely that we arrive at equations of the form

$$\frac{d}{dt}\begin{pmatrix} F^+ \\ F^- \end{pmatrix} = \begin{pmatrix} t & -r \\ r & -t \end{pmatrix}\begin{pmatrix} F^+ \\ F^- \end{pmatrix} + \begin{pmatrix} Q^+ \\ Q^- \end{pmatrix} \tag{15.15}$$

(a) The Two-Stream Equations—The Conceptual Approach

The arguments formulated here are similar to those used in the pioneering work on radiative transfer by Schuster in 1905. Consider a parallel, horizontally uniform slab of cloud and consider the fluxes flowing in two opposing directions.[3] We will use the + superscript to refer to quantities associated with flow in the upward direction and a – superscript on quantities relevant to downward flow. The two-stream equations define the energy balance of this thin slab of thickness Δz in exactly the same way as Eqn. (15.6b) describes an energy balance of a small volume of cloud. In order to express the radiative energy budget of a layer Δz thick, it is necessary to define the following optical properties:

- The proportion of the incident flux lost by absorption as the radiation flows through the layer of unit thickness is $k_{abs}D^{\pm}$ where $D^{\pm}$ is a measure of the 'diffuseness' of the radiation field. This parameter more or less represents the mean extension of the path, relative to the vertical, that a diffuse radiation field travels as it penetrates the layer. It is a function of the angular properties of the intensity field among other parameters and represents one of the simplifications mentioned above. If we suppose that the angular distribution of radiation that produces the flux is the same in both directions (the magnitudes might be different), then

$$D^+ = D^- .$$

Although this assumption is questionable, it tends to be universally used in two-stream models.

- The proportional loss of flux by scattering is $s_{sca}b^{\pm}$ per unit thickness. Here we note that the process of absorption is treated differently from scattering in that a measure of the path length is needed for estimating absorption but this measure is not needed for scattering. We will further suppose that this scattering is the same whether the radiation flows upward or downward, and thus

$$b^+ = b^- .$$

Another parameter of relevance is the fraction of radiation f that is scattered in the forward direction. This fraction is defined such that

$$f + b = 1 \tag{15.16}$$

For a change in flux ΔF defined as positive upwards, then the change in flux on transfer through the layer Δz is

$$\Delta F^{\pm} = \mp(Dk_{abs} + s_{sca}b)F^{\pm}\Delta z \pm s_{sca}bF^{\mp}\Delta z(\pm Q^{\pm}\Delta z) \tag{15.17}$$

[3] The relationship between radiative flux and intensity is explored in the Appendix. The derivation of the two-stream equations given here follows the more conceptual arguments of Schuster. The same equations can be derived directly from Eqn. (15.6b) given some assumption about the intensity field. This alternative derivation is left for later.

where the last term in parentheses represents internal sources of $F^{\pm}$ in the layer Δz^4. The first two terms on the right hand side and enclosed by parentheses describe the losses of radiation through the processes of absorption and scatter, respectively, while the middle terms represent the increase of flux by backscatter of the opposing stream. Introducing the definition of optical thickness as

$$\Delta\tau = -(k_{abs} + s_{sca})\Delta z$$

where the minus sign defines τ as increasing downwards from cloud base opposite to the change in z. On taking the limit $\Delta z \rightarrow 0$, we obtain the two-flow radiative transfer equation

$$\mp\frac{dF^{\pm}}{d\tau} = -[D(1-\tilde{\omega}_o) + \tilde{\omega}_o b]F^{\pm} + \tilde{\omega}_o bF^{\mp}(+Q^{\pm}) \tag{15.18}$$

where $\tilde{\omega}_o = s_{sca}/(s_{sca} + k_{abs})$. All two-stream methods described in the literature essentially reduce down to this equation. The only difference between the various methods lies in how D, b, and $S^{\pm}$ are specified. One example is to consider the simple phase function introduced in Section 13.7a, then it follows that

$$b = (1 - g)/2$$

where g is the phase function asymmetry. The radiative transfer equation then becomes

$$\mp\frac{dF^{\pm}}{d\tau} = -\left[D(1-\tilde{\omega}_o) + \frac{\tilde{\omega}_o}{2}(1-g)\right]F^{\pm} + \frac{\tilde{\omega}_o}{2}(1-g)F^{\mp}(+Q^{\pm})\,. \tag{15.19}$$

The general solution to Eqn. (15.19) for given sources can be complicated. Here we neglect this term and consider only solar radiation incident on cloud top assuming this incident flux is purely diffuse (as opposed to the more realistic case of a purely collimated incident flux). While the details of the solutions described below change with the addition of the source term for solar radiation, notably by introducing a solar zenith angle dependence to the solutions, the gross relationships between the optical properties of clouds (τ^*, $\tilde{\omega}_o$, and g) and the diffuse reflectance and transmittance does not change.

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[4] Two main sources of flux are usually considered in these models. One is the source of radiation due to thermal emission, which according to Kirchoffs law takes the form

$$Q^{\pm} = k_{abs}\pi B(T)$$

for emitting cloud particles of temperature T. The second is the source of diffuse radiation that results from the single scattering of a collimated flux $F_{\odot}$ of sunlight. This source has the form

$$Q^{\pm} = F_{\odot}e^{-\tau/\mu_{\odot}}s_{sca}\begin{pmatrix} b_{\odot} \\ f_{\odot} \end{pmatrix}$$

where $f_{\odot}$ and $b_{\odot}$ are the forward and backward scattering fractions of the incident flux $F_{\odot}$ and these fractions are functions of the cosine of the solar zenith angle $\mu_{\odot}$

Example 15.3: Solution for sourceless atmosphere, pure scattering

Consider the example of a single layer of 'cloud' with S = 0. For pure scattering, $\tilde{\omega}_o = 1$, $k_{abs} = 0$

$$F^{\pm} = m_+ \mp m_-(1 + \tilde{\tau})$$

where m_+ and m_- are constants determined by boundary conditions and

$$\tilde{\tau} = (1 - g)\tau^*$$

is the optical depth of the entire slab, τ*, scaled by the factor (1 - g). The relevance of this scaled parameter becomes apparent by considering an isolated scattering and absorbing layer illuminated from above by flux $F_\odot$ overlying a dark surface. Under these conditions, the albedo of the cloud layer is

$$R = \frac{F^+(0)}{F_\odot} = \frac{\tilde{\tau}}{2 + \tilde{\tau}} \qquad (15.20a)$$

and the transmittance

$$T = \frac{F^-(\tau^*)}{F_\odot} = 1 - R = \frac{2}{2 + \tilde{\tau}} \qquad (15.20b)$$

This result implies that two non-absorbing cloud layers with different optical thicknesses τ* and g reflect the same amount of radiation when the respective values of $\tilde{\tau}$ are the same. This is referred to as a similarity condition and implies that it is not possible to infer τ* from a single reflection or a single transmission measurement without information about g. One of the problems associated with the remote sensing of ice crystal clouds is that g is neither well known nor well understood in how it varies with different crystal habits. This parameter is well known for water droplet clouds and is quasi-constant with a typical value in the range 0.8-0.85.

Example 15.4: Solution for sourceless atmosphere, nonconservative scattering

For this case, $\tilde{\omega}_o < 1$, $k_{abs} > 0$: The solution to Eqn. (15.19) for a sourceless, uniform medium has the form

$$F^{\pm}(\tau) = m_+ \gamma_{\pm} e^{kt} + m_- \gamma_{\mp} e^{-kt} \qquad (15.21a)$$

where

$$k = \{(1-\tilde{\omega}_o)D[(1-\tilde{\omega}_o)D + 2\tilde{\omega}_o b]\}^{1/2} \qquad (15.21b)$$

and

$$\gamma_{\pm} = 1 \pm (1-\tilde{\omega}_o)D/k \qquad (15.21c)$$

where as above the coefficients $m\pm$ are determined from appropriate boundary conditions. Consider the same conditions applied to Eqns. (15.21a,b)

$$F^+(\tau^*) = 0$$

$$F^-(0) = F_\odot$$

for an isolated layer of optical thickness τ^*. With some manipulation of Eqn. (15.21a), the albedo and transmittance of the layer can be written as

$$R = \gamma_+\gamma_-[e^{-k\tau^*} - e^{-k\tau^*}]/\Delta(\tau^*) \qquad (15.22a)$$

$$T = (\gamma_+^2 - \gamma_-^2)/\Delta(\tau^*) \qquad (15.22b)$$

where

$$\Delta(\tau^*) = \gamma_+^2 e^{k\tau^*} - \gamma_-^2 e^{-k\tau^*} \qquad (15.22c)$$

As $\tau^* \rightarrow \infty$, $R \rightarrow R_\infty = \gamma_-/\gamma_+$ and this is referred to as the albedo of a semi-infinite cloud. This represents the upper limit to the albedo of a cloud and since $T = 0$ and $A = 1 - R$, this is also the upper limit to the absorption A_∞ within the cloud. These upper limits are determined entirely by the optical properties $\tilde{\omega}_o$, k, g, and D of the cloud. From the substitution of Eqn. (15.22b) in Eqn. (15.22c) together with the definition of R_∞, it follows that

$$R_\infty = \frac{[1+1/s^2]^{1/2} - \sqrt{2}}{[1+1/s^2]^{1/2} + \sqrt{2}} \qquad (15.23)$$

where

$$s = \left(\frac{1-\tilde{\omega}_o}{1-\tilde{\omega}_o g}\right)^{1/2}$$

is another similarity parameter (Fig. 15.4b). Equation (15.23) states that the reflection by two different optically thick clouds are equivalent when the similarity parameter s is equivalent.

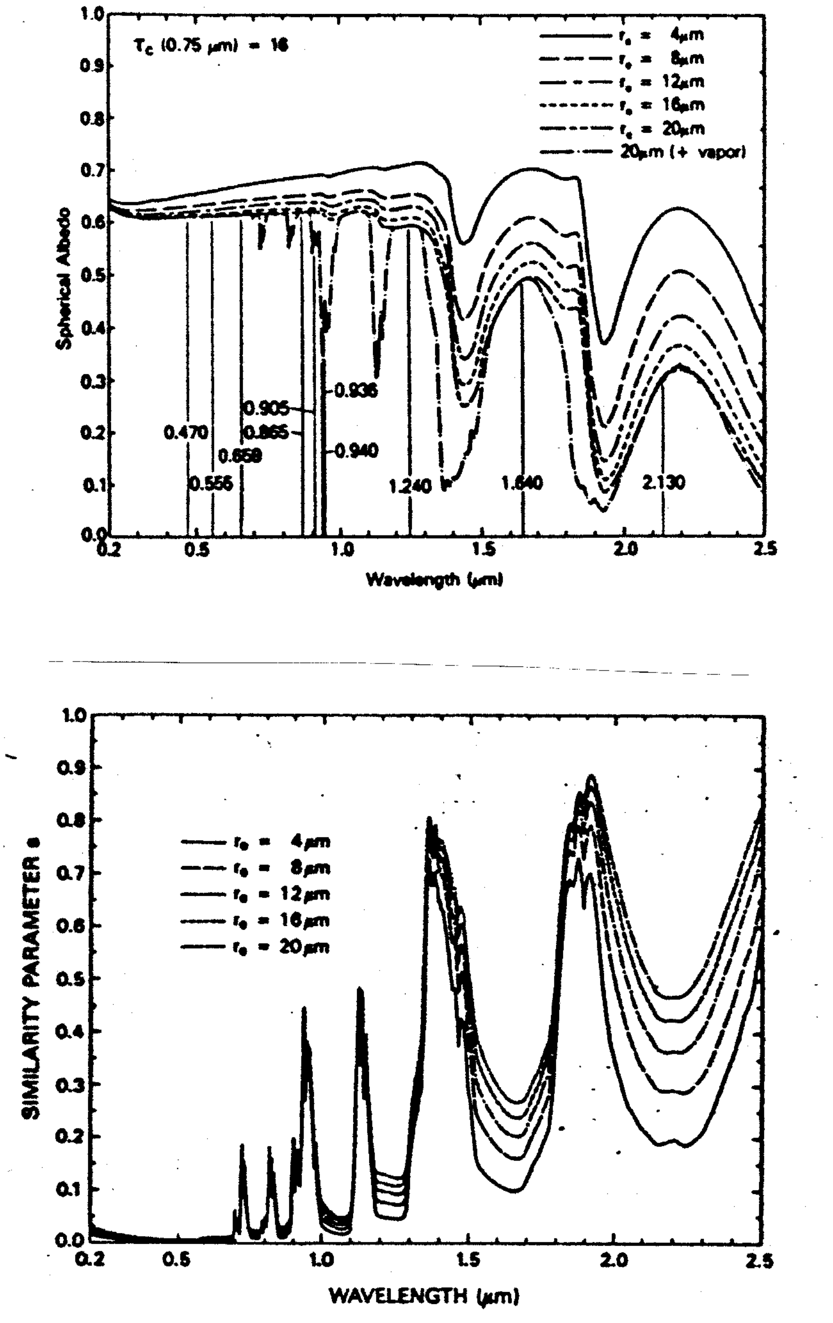

Fig. 15.4 *(a) The spectral reflectance from modeled clouds as a function of their particle size. (b) The similarity parameter as a function of wavelength for different assumed values of the cloud droplet effective radious* r_e.

Example 15.5: Pollution Susceptible Clouds

The effect of ship stack effluents on cloud optical depth and cloud albedo is a topic of intensive interest. The simple two-stream model introduced previously now serves to emphasize how the scaled optical depth is the direct controlling parameter on the albedo of clouds. We can deduce that optical depth of clouds is

$$\tau^* \approx 2\pi N_o \bar{r}^2 h$$

for a cloud of depth h composed of N_o particles of a size $\bar{r}$ that exceeds the wavelength of radiation. Increased water content (occurring largely as an increase in $\bar{r}$), for instance, can increase the optical depth of clouds. An increase in number concentration N_o can also increase τ^* and the sensitivity of optical thickness τ^* to N_o, for constant liquid-water content, is given by

$$\frac{\Delta\tau^*}{\tau^*} = \frac{1}{3}\frac{\Delta N_o}{N_o}.$$

For cloud droplets under solar illumination, g is quasi-constant and ≈ 0.85. Using this value in Eqn. (15.20a), one obtains the following simple approximate expression

$$R \cong \frac{\tau}{13+\tau}.$$

for the albedo of a cloud. We can readily derive the sensitivity of R to droplet number N_o from this relation and express it in terms of N_o and R. The result for fixed liquid water content w is

$$\left(\frac{dR}{dN_o}\right)_w = \frac{R(1-R)}{3N_o}.$$

Thus, for a given N_o, the most susceptible clouds are those with $R \approx 1/2$, but the maximum of R is rather flat - for R = 1/4 or 3/4, dR/dN_o is still three-fourths of its maximum value. For fixed R, $(dR/dN_o)_w$ is inversely related to N_o, which in the real present atmosphere, can vary by more than two orders of magnitude. The susceptibility $dRldN_o$ (graphed in Fig. 15.5) reveals a considerable sensitivity for clean conditions—e.g., in oceanic and remote areas (where N_o is low). There $(dR/dN_o)_w$ *is* seen to approach 1% (per cm^{-3}); that value would mean a reflectance change *of* 0.01 for a concentration change of just 1 cm^{-3}.

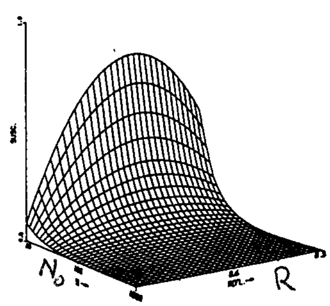

Fig. 15.5 Susceptibility for different conditions of N_o and R.

b) The Two-Stream Equations—Analytic Approach: Eddington's Approximation as an Example

Suppose we consider azimuthally averaged quantities $\overline{I}$,

$$F = 2\pi \int_0^1 \overline{I}\,\mu d\mu$$

then

$$\mu d\frac{\overline{I}}{d\tau} = \overline{I} - \frac{\tilde{\omega}_o}{2}\int_{-1}^{+1} \overline{p}(\mu,\mu')\overline{I}(\mu')d\mu'$$

- If we assume that

$$\overline{I} \approx I_o + I_1\mu$$

which resembles the form used in our diffusion approximation, then

$$F^+ = 2\pi\int_0^1 I\mu d\mu = 2\pi\left[\frac{I_o}{2} + \frac{1}{3}I_1\right]$$
$$F^- = 2\pi\int_0^{-1} I\mu d\mu = 2\pi\left[-\frac{1}{3}I_1 + \frac{I_o}{2}\right]$$

and

$$F_{net} = F^+ - F^- = 2\pi\int_{-1}^{+1} I\mu d\mu = \left(\frac{2}{3}I_1\right)2\pi$$
$$I_1 = \frac{3}{4\pi}(F^+ - F^-)$$
$$F^+ + F^- = 2\pi I_o$$
$$I_o = \frac{F^+ + F^-}{2\pi}$$

- The second approximation we introduce is the following

$$\overline{P}(\mu,\mu') = 1 + 3g\mu\mu'$$

for the phase function expansion. If we consider our radiative transfer equation and integrate over each respective hemisphere, then we obtain the following two equations (ignoring sources for the moment):

$$2\pi\int_0^1 \mu\underbrace{\frac{dI}{d\tau}(\mu)}_{A}d\mu = 2\pi\int_0^1 I\underbrace{(\mu)}_{B}d\mu - \frac{\tilde{\omega}_o}{2}\int_0^1\int_{-1}^{+1}\underbrace{p(\mu,\mu')I(\mu')}_{C}d\mu' d\mu$$
$$2\pi\int_0^{-1} \mu\underbrace{\frac{dI}{d\tau}(\mu)}_{A'}d\mu = 2\pi\int_0^{-1} I\underbrace{(\mu)}_{B'}d\mu - \frac{\tilde{\omega}_o}{2}\int_0^{-1}\int_{-1}^{+1}\underbrace{p(\mu,\mu')I(\mu')}_{C'}d\mu' d\mu$$

Consider the first equation:

Term $A = 2\pi\int_0^1 \mu \frac{dI}{d\tau} d\mu = 2\pi \frac{d}{d\tau}\int_0^1 \mu[I_o + I_1\mu]d\mu = \frac{d}{d\tau}F^+$

Term $A' = \frac{d}{d\tau}F^-$

Term $B = 2\pi\int_0^1 (I_o + I_1\mu)d\mu = (I_o + \frac{1}{2}I_1)2\pi$

Term $B' = 2\pi\int_0^{-1} (I_o + I_1\mu)d\mu = (\frac{1}{2}I_1 - I_o)2\pi$

Term $\frac{C}{2\pi} = \frac{\tilde{\omega}_o}{2}\int_0^1 d\mu \int_{-1}^{+1} (I_o + I_1\mu') + 3g\mu(I_o\mu' + I_1\mu'^2)d\mu'$

$= \frac{\tilde{\omega}_o}{2}\int_0^1 d\mu [I_o\mu' + \frac{1}{2}I_1\mu'^2 + \frac{3}{2}g\mu I_o\mu'^2 + g\mu I_1\mu'^3]_{-1}^{+1}$

$= \frac{\tilde{\omega}_o}{2}\int d\mu[2I_o + 2g\mu I_1]$

Term $\frac{C(C')}{2\pi} = \tilde{\omega}_o \int_0^{1(-1)} d\mu(I_o + g\mu I_1)$

$= \tilde{\omega}_o[I_o + \frac{1}{2}gI_1](\tilde{\omega}_o[\frac{1}{2}gI_1 - I_o])$

After collecting terms, we obtain

$$\frac{dF^+}{d\tau} = 2\pi\left(I_o + \frac{1}{2}I_1\right) - 2\pi\tilde{\omega}_o\left[I_o + \frac{1}{2}gI_1\right]$$

$$= F^+ + F^- + \frac{3}{4}(F^+ - F^-) - \tilde{\omega}_o\left[F^+ + F^- + \frac{3}{4}g(F^+ - F^-)\right]$$

$$= \frac{7}{4}F^+ - \frac{\tilde{\omega}_o}{4}(4+3g)F^+ + \frac{1}{4}F^- - \frac{\tilde{\omega}_o}{4}(4-3g)F^-$$

or

$$\frac{dF^+}{d\tau} = \left\{\frac{7}{4} - \frac{\tilde{\omega}_o}{4}(4+3g)\right\}F^+ + \left\{\frac{1}{4} - \frac{\tilde{\omega}_o}{4}(4-3g)\right\}F^-$$

Similarly

$$\frac{dF^-}{d\tau} = 2\pi\left[-I_o + \frac{1}{2}I_1\right] = 2\pi\tilde{\omega}_o\left[\frac{1}{2}gI_1 - I_o\right]$$

$$= \left[-F^+ - F^- + \frac{3}{4}(F^+ - F^-)\right] - \tilde{\omega}_o\left[-F^+ - F^- + \frac{3}{4}g(F^+ - F^-)\right]$$

$$= -\frac{7}{4}F^- + \frac{\tilde{\omega}_o}{4}(4+3g)F^- - \frac{1}{4}F^+ + \frac{\tilde{\omega}_o}{4}(4-3g)F^+$$

or

$$\frac{dF^-}{d\tau} = -\left\{\frac{7}{4} - \frac{\tilde{\omega}_o}{4}(4+3g)\right\}F^- - \left\{\frac{1}{4} - \frac{\tilde{\omega}_o}{4}(4-3g)\right\}F^+$$

where we can readily identify

$$\begin{aligned} t &= \frac{7}{4} - \frac{\tilde{\omega}_o}{4}(4+3g) \\ r &= -\frac{\tilde{\omega}_o}{4}(4-3g) \end{aligned} \tag{15.24}$$

(c) Delta-Two Stream Models

We have already remarked on the scaling associated with phase functions of the form

$$p(\mu,\mu') = 2f\delta(\mu'-\mu) + (1-f)(1+3g\mu\mu')$$

which reduces to the formal two-stream solutions when

$$\begin{aligned} g' &= \frac{g-f}{1-f} \\ \tau' &= (1-\tilde{\omega}_o f)\tau \\ \tilde{\omega}_o' &= \frac{(1-f)\tilde{\omega}_o}{1-f\tilde{\omega}_o} \end{aligned}$$

are used directly in the solutions. It is usual to employ the second moment of the expansion, namely

$$f = \chi/5 = g^2$$

for the scaling factor.

15.4 General Solutions

The two-stream model and its general solution are briefly introduced here. We will consider two kinds of source functions to represent those described in footnote 2. In developing these solutions, it is useful to introduce two-stream equations (Eqn. (15.15)) as follows

$$L_F F = Q \tag{15.25a}$$

where F is a flux vector

$$F = \begin{pmatrix} F^+ \\ F^- \end{pmatrix}$$

of upwelling (F^+) and downwelling (F^-) flux at level z'. The dependence of each factor in Eqn. (15.25a) on z' is taken to be understood. The source function vector,

$$Q = \begin{pmatrix} Q^+ \\ Q^- \end{pmatrix}$$

too depends on z'. The two-stream transport operator is

$$L_F = \frac{d}{dz'} - \sigma_{ext} \begin{pmatrix} -t & r \\ -r & t \end{pmatrix} \tag{15.25b}$$

where we note the streaming term is defined relative to z rather than τ as in Eqn. (15.15), which means that the coefficients t and r differ from those of Eqn. (15.24) only by a factor of σ_{ext}. Although these define the flux equations, the form of this equation is generic in the sense that they also equally apply to radiance and the 'n stream' problem (e.g., Flatau and Stephens, 1988).

The different forms of the equation coefficients in Eqn. (15.15), namely t and r, define different version of a two-stream model. The 2 x 2 matrix of coefficients defines the attenuation matrix

$$\mathbf{A} = \sigma_{ext} \begin{pmatrix} -t & r \\ -r & t \end{pmatrix} \tag{15.26}$$

and the 'solution' to the sourceless equation (i.e., Q = 0) can be expressed in terms of the matrix exponential

$$\mathbf{M}(z, y) = e^{-\mathbf{A}(z-y)} \tag{15.27}$$

where **M** is a mapping function. By virtue of the block structure of **A**, this mapping matrix has a similar form

$$\mathbf{M}(z, y) = \begin{pmatrix} m_{++}(z, y) & m_{+-}(z, y) \\ m_{-+}(z, y) & m_{--}(z, y) \end{pmatrix} \tag{15.28}$$

For the 2 x 2 matrix **A** of the two-stream equations, Eqn. (15.27) follows as

$$m_{++}(z, y) = \frac{e^{\kappa(z-y)}}{2} f_+ + \frac{e^{-\kappa(z-y)}}{2} f_-$$
$$m_{+-}(z, y) = \frac{r}{\kappa}\left[\frac{e^{-\kappa(z-y)}}{2} - \frac{e^{\kappa(z-y)}}{2}\right]$$

$$m_{-+}(z,y)=\frac{r}{\kappa}\left[\frac{e^{\kappa(z-y)}}{2}-\frac{e^{-\kappa(z-y)}}{2}\right]$$
$$m_{--}(z,y)=\frac{e^{\kappa(z-y)}}{2}f_{-}+\frac{e^{-\kappa(z-y)}}{2}f_{+} \qquad (15.29)$$

where

$$f_{+}=(1+t/\kappa)$$
$$f_{-}=(1-t/\kappa)$$

(a) The Interaction Principle

Consider the two types of radiative transfer problem as posed in Fig. 15.6. The goal of the first is to deduce the fluxes at the upper boundary of an isolated layer at z_T in terms of the fluxes at the lower boundary z. Stated, this way the radiative transfer problem is an initial value problem. Its solution is as follows. First consider the sourceless equations for which the solution (assuming constant coefficients) is

$$\begin{pmatrix}F^{+}\\F^{-}\end{pmatrix}(z)=\begin{pmatrix}m_{++}(z,y) & m_{+-}(z,y)\\ m_{-+}(z,y) & m_{--}(z,y)\end{pmatrix}\begin{pmatrix}F^{+}\\F^{-}\end{pmatrix}(y) \qquad (15.30)$$

Unfortunately, most problems of radiative transfer are posed as follows. Given fluxes incident on the boundaries, what are the emergent fluxes (Fig. 15.6). These are two point boundary value problems, which can be solved through rearrangement of Eqn. (15.30). The relationship between fluxes out in terms of fluxes in (and internal sources) is referred to as the interaction principle. In rearranging Eqn. (15.30) in the form of the interaction principle, we obtain the relation between the mapping functions above and the more classical properties of reflection and transmission.

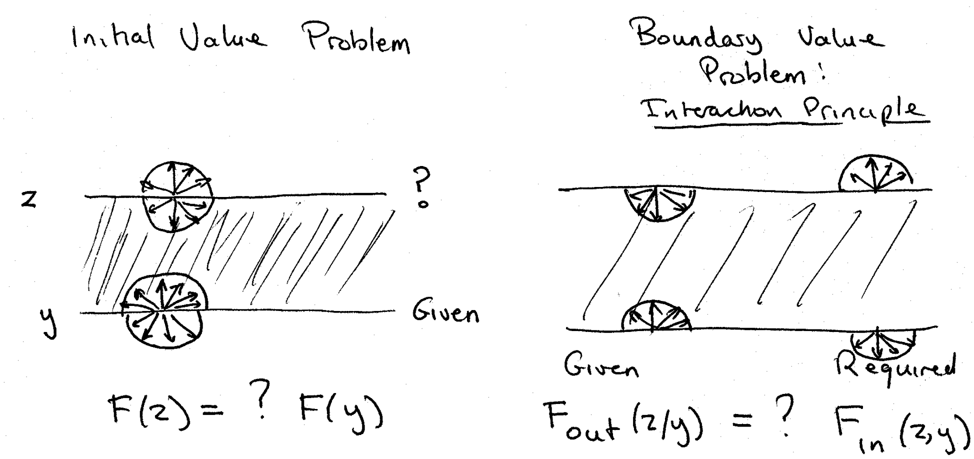

Fig. 15.6 Two types of transfer problems, the initial value problem at left and the more traditional two-point boundary value problem defining the interaction principle.

Simple reorganization of Eqn. (15.30) in its interaction form gives the desired emergent fluxes $F^+(z_T)$ and $F^-(z)$ in terms of input fluxes

$$\begin{pmatrix} F^+(z_T) \\ F^-(z) \end{pmatrix} = \begin{pmatrix} 1/m_{++} & -m_{+-}/m_{++} \\ m_{-+}/m_{++} & m_{--} - m_{-+}m_{+-}/m_{++} \end{pmatrix} \begin{pmatrix} F^+(z) \\ F^-(z_T) \end{pmatrix}. \tag{15.31}$$

where the notation indicating the mapping factors are defined for the layer (z_T, z) is dropped for convenience. This identifies the layer diffuse reflection and transmission functions as

$$R(z_T, z) = \frac{-m_{+-}(z_T, z)}{m_{++}(z_T, z)} \tag{15.32a}$$

$$T(z_T, z) = \frac{1}{m_{++}(z_T, z)} \tag{15.32b}$$

which are those of Eqn. (15.22).

(b) Adding Sources - General Solution

We proceed with Eqn. (15.25b) in Eqn. (15.25a) and multiplying both sides by the exponential of the matrix

$$e^{-Az'}\frac{dF}{dz'} - e^{-Az'}AF(z') = e^{-Az'}Q(z') \tag{15.33}$$

where we assume that the attenuation matrix (i.e., the optical properties r and t) is independent of z'. Integration of Eqn. (15.33) from $(z \rightarrow z_T)$ yields

$$F(z) = e^{-A(z_t - z)}F(z_T) + S(z_T, z) \tag{15.34}$$

by virtue of the property of the matrix exponential

$$[e^{-Az}]^{-1} = e^{Az}$$

and where the vector

$$S(z_T, z) = -\int_z^{z_T} e^{-A(z_t - z)}Q(z')dz' \tag{15.35}$$

These resemble the more traditional integral form of the radiative transfer equation (Sections 4 and 10). However, it contains the desired emergent fluxes (i.e., the solution) on both sides of the equation as seen more clearly in the expanded form

$$\begin{pmatrix} F^+(z) \\ F^-(z) \end{pmatrix} = \begin{pmatrix} m_{++}(z_T, z) & m_{+-}(z_T, z) \\ m_{-+}(z_T, z) & m_{--}(z_T, z) \end{pmatrix} \begin{pmatrix} F^+(z_T) \\ F^-(z_T) \end{pmatrix} + \begin{pmatrix} S^+(z_T, z) \\ S^-(z_T, z) \end{pmatrix}. \tag{5.36}$$

A special solution arises for problems in which the medium is illuminated with zero incident fluxes (known as vacuum boundary conditions). Then we obtain

$$\begin{pmatrix} F^+(z_T) \\ F^-(z) \end{pmatrix} = \begin{pmatrix} -S(z_T,z)/m_{++}(z_T,z) \\ -m_{-+}(z_T,z)S^+(z_T,z)/m_{++}(z_T,z)+S^-(z_T,z) \end{pmatrix} \tag{15.37}$$

which are the particular solutions to Eqn. (15.25a) for general solutions.