

# AT622 Section 9

## Models of Transmission

The aim of this section is to introduce popular techniques used to model transmission through an absorbing layer of gas. We have already seen how the mathematical description of absorption by the gases of the atmosphere can be formulated in terms of transmission functions (Section 4). There are several types of transmission functions that must be learned and the connections to one another understood. The ultimate purpose, however is to be able to characterize the transmission averaged over many absorption lines (band transmission). Connections between different forms of transmission functions are shown in Fig. 9.1. There are transmission functions that apply to the transmission of intensity and the transmission of flux. Transmission can either be monochromatic (i.e., at a single wavelength) or broadband (i.e., an average over a band of several wavelengths), which is one of the goals of this section. These functions can either apply to homogeneous paths (i.e., applies to uniform path of fixed  $p$  and  $T$  such as encountered in the laboratory measurements) or to heterogeneous paths of varying  $p$  and  $T$  (such as in the atmosphere). We will see that the transformation from intensity to flux transmission is largely trivial and we will spend the most time discussing broadband transmission models and how we treat absorption along variable  $p$  and  $T$  paths.

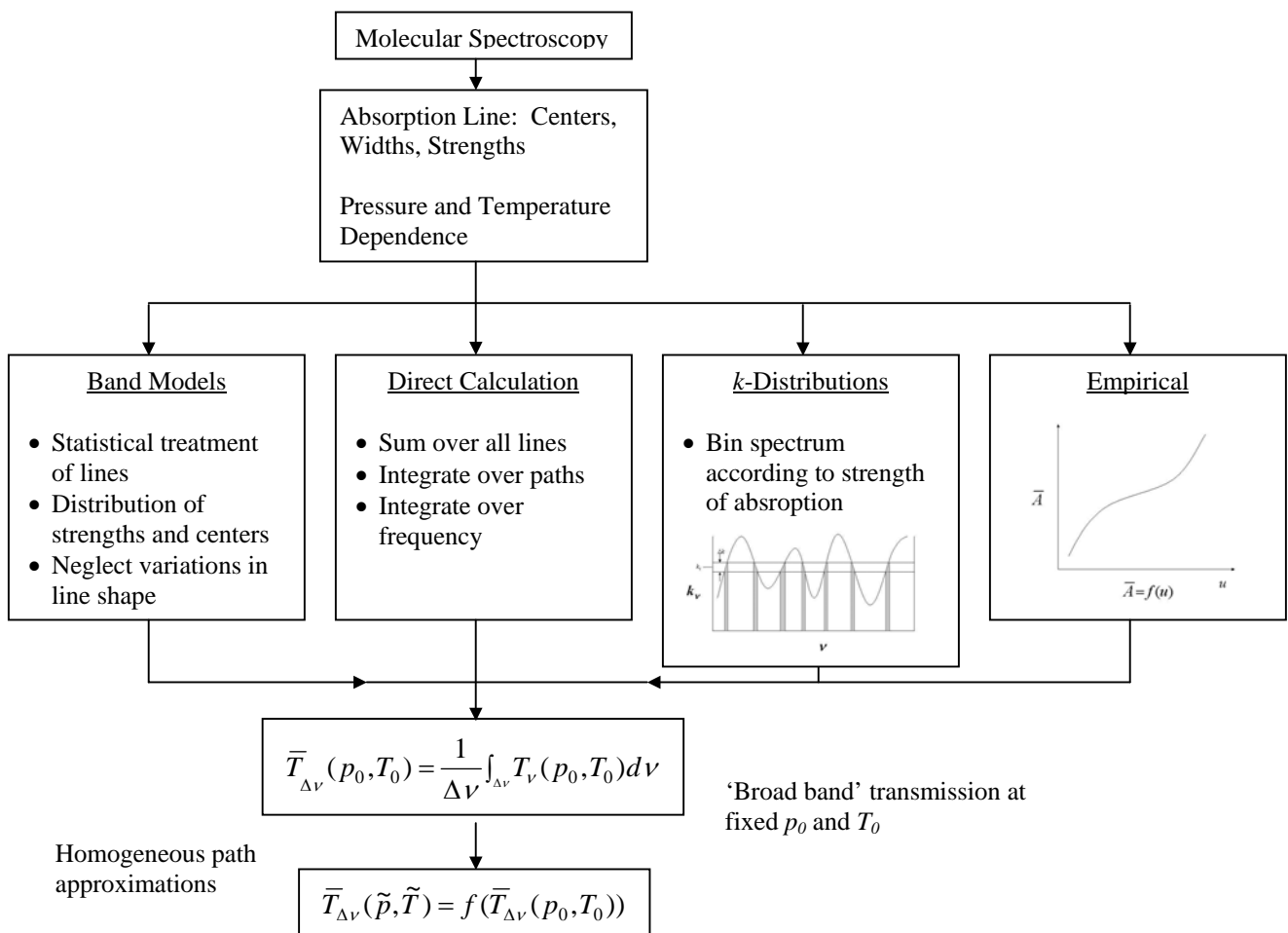


Fig. 9.1 Connections between different forms of transmission functions.

## 9.1 Basic Definition of Transmission Functions

The exchange of radiation with the gases of the atmosphere is described in terms of a transmission function. The concept of transmission follows directly from Lambert's law of extinction.

$$dI_\nu = -k_\nu I_\nu ds \quad (9.1)$$

where  $k_\nu$  is the absorption coefficient (here we consider only absorption and ignore scattering),  $I_\nu$  the intensity of the radiation field, and  $ds$  is some measure of path, defined such that the quantity

$$d\tau_\nu = k_\nu ds$$

is unitless. Solution of Eqn. (9.1) gives

$$I_\nu(\tau_\nu) = I_\nu(\tau_\nu = 0)e^{-\tau_\nu} \quad (9.2)$$

where

$$\mathcal{T}_\nu = e^{-\tau_\nu} = 1 - \mathcal{A}_\nu \quad (9.3)$$

is the monochromatic "transmission" function and  $\mathcal{A}_\nu$  is the monochromatic absorption. Since the relationship between absorption and transmission is trivial, we will develop our models in terms of either absorption or transmission.

The absorption coefficient (and for that matter the scattering coefficient) can be defined in a number of different ways according to how we measure the amount of matter along the path. Table 9.1 gives four more commonly used quantities together with the specification of the amount of matter. Note again that the product of extinction coefficient and amount of matter is unitless.

Table 9.1 provides the conversion factors between the different forms of extinction coefficient (read extinction here as absorption). With volume extinction, the computations use distance as the independent variable. This is generally only used in calculations involving particle absorption (and scattering). For gases, the path length is usually defined in a way that reflects gaseous density and it is more convenient to use one of the other forms. Of these,  $e_s$  is the most popular choice by many spectroscopists and thus we see the path lengths per centimeter at STP often used in empirical transmission formulas like those presented later.

Table 9.1 Dimensions and conversion factors for extinction coefficients<sup>a</sup>.

Symbol	$e_\nu$	$e_m$	$e_n$	$e_s$
Name	Volume e.c.	Mass e.c.	Molecular e.c.	e.c. per cm s.t.p.
Dimensions	cm <sup>-1</sup>	g <sup>-1</sup> cm <sup>2</sup>	cm <sup>2</sup>	cm <sup>-1</sup>
$e_\sigma$	1	$\rho^{-1}$	$n^{-1}$	$n_s/n$
$e_m$	$\rho$	1	$m$	$\rho_s$
$e_n$	$n$	$m^{-1}$	1	$n_s$
$e_s$	$n/n_s$	$\rho_s^{-1}$	$n_s^{-1}$	1

<sup>a</sup>  $\rho$  = density of absorbing gas (g cm<sup>-3</sup>)       $n_s$  = molecular no. density at s.t.p. (Loschmidt no., cm<sup>-3</sup>)  
 $\rho_s$  = density of absorbing gas at s.t.p. (g cm<sup>-3</sup>)       $m$  = molecular mass (g)  
 $n$  = molecular number density (cm<sup>-3</sup>)

**Example 9.1:** Two specific examples of conversion highlight the advantage of  $e_n$  and the number of molecules per centimeter as the measure of attenuation gas over the other combinations. From Table 9.1 we deduce that

$$1 \text{ cm of gas at STP} = \frac{M}{2.24 \times 10^4} \text{ gcm}^{-2},$$

where  $M$  is the molecular weight of the gas

$$1 \text{ cm - STP} = 2.69 \times 10^{19} \text{ molecules cm}^{-2}.$$

The above expression is valid for all gases. It follows from these two examples that for water vapor

$$1 \text{ g cm}^{-2} (\text{H}_2\text{O}) = 3.34 \times 10^{22} \text{ molecules cm}^{-2}.$$

Thus, the unit “molecule  $\text{cm}^{-2}$ ” is independent of the nature of the absorbing gas and basic to all gases and offers a way of unifying absorber concentration units for all atmospheric constituents.

Despite this benefit, absorption by gases is often expressed in terms of the mass absorption coefficient. For this case, the path element is expressed as

$$du = \rho_g ds$$

and in terms of the mixing ratio (mass)

$$r = \rho_g / \rho_{air}$$

$$du = r \rho_{air} ds$$

For vertical paths (together with the hydrostatic approximation), it follows that

$$u(z_1, z_2) = \int_{z_1}^{z_2} r \rho_{air} dz = \frac{1}{g} \int_{p_2}^{p_1} r dp \quad (9.4)$$

This is a formula that should be learned and its derivation understood.

**Example 9.2:** Water vapor and sea surface temperature. Develop a relationship between the vertically integrated water vapor path through the entire vertical extent of the atmosphere (precipitable water) and the sea surface temperature. Assume

- (a) the vertical profile of specific humidity (expressed in terms of mass mixing ratio) has the following form  $r(p) = r_s(p/p_s)^\lambda$  where  $r_s$  is the surface mixing ratio.
- (b) the saturation vapor pressure at the surface is  $e_s \approx be^{[a(T_s - T_o)]}$  such that  $r_s \approx rh0.622e_s/p_s$ .

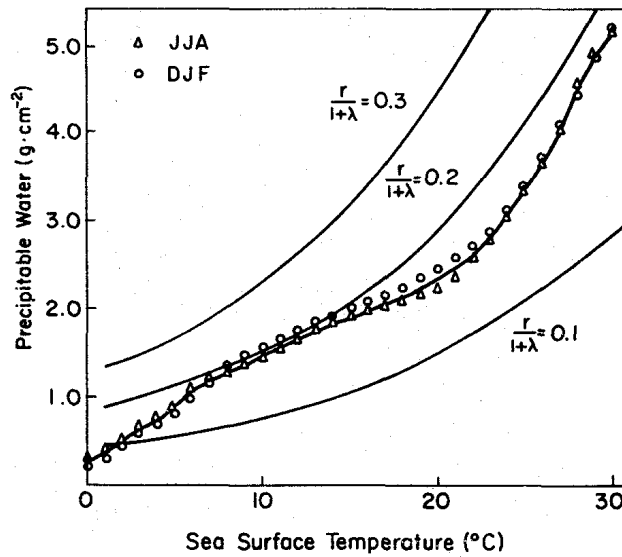
Derive your answer in terms of the surface relative humidity,  $\lambda$  and the SST.

**Answer:** The column path follows from Eqn. (9.4) as

$$u = \frac{rh}{gp_s} be^{[a(T_s - T_o)]} \int_0^{p_s} (p/p_s)^\lambda dp$$

or

$$u = 0.622b \left( \frac{rh}{g(1 + \lambda)} \right) e^{a(T_s - T_o)}$$



## 9.2 From Intensity (Beam) to Flux (Diffuse) Transmission

Transmission along a slant path  $s_1 \rightarrow s_2$

$$T(s_1, s_2) = e^{-\int k_m dz / \mu} = T(z_1, z_2, \mu)$$

$Tr(z_1, z_2, \mu)$  is referred to as the beam (or intensity) transmission function for the path defined by  $(z_1, z_2, \mu)$ .

The flux transmission function is defined as (assume azimuthal symmetry)

$$\mathcal{T}_f(z_1, z_2) = \int_0^1 \mu T(z_1, z_2, \mu) d\mu / \int_0^1 \mu d\mu \quad (9.5)$$

which is a  $\mu$ -weighted transmission function characterizing the transmission of the irradiance (flux) through the slab  $z_1 \rightarrow z_2$ .

As noted previously in section 4, we can write Eqn. (9.5) in the following way

$$\mathcal{T}_f(z_1, z_2) = 2E_3[\tau(z_1, z_2)] \quad (9.6)$$

where  $E_3(x)$  is the  $n^{\text{th}}$  exponential integral

$$E_n(x) = \int_1^\infty e^{-\eta x} d\eta / \eta^n$$

( $\eta = 1/\mu, x = \tau$ ). To a high degree of accuracy,

$$2E_3(x) \approx e^{-\beta x} \quad (9.7)$$

where  $\beta = 1.66$  (the so called diffusivity factor). Therefore,

$$\mathcal{T}_f(z_1, z_2) = e^{-\beta \int k du} \quad (9.8)$$

The important point here is that the flux transmission can be modeled using the transmission for intensity with the path merely increased by the diffusivity factor  $\beta$ . Thus in developing theories for broadband functions, we will consider intensity transmission and note that broadband flux transmission is given by this transmission function with the introduction of this diffusivity function.

### 9.3 Frequency Integrated Absorption of a Single Line

Most problems of interest require spectrally integrated transmission (or equivalently absorption) functions over a variety of spectral scales varying from the scale defined by the line half width to scales attached to broad spectral regions 10's-100's  $\text{cm}^{-1}$  wide. Before understanding how we can do this complicated integration, it is useful to study the heuristic properties of the integrated absorption of a single line.

The quantity of main interest is the monochromatic absorption as defined by the frequency integrated absorption, namely

$$W(u) = \int d\nu (1 - e^{-k_\nu u}) = \int d\nu (1 - e^{-Sf(\nu-\nu_o)u}) \quad (9.9a)$$

where  $u$  replaces  $s$  as a symbol of the measure of path. This absorption is called the equivalent width  $W(u)$  since it measures the width of  $\nu$  units of a hypothetical square shaped line that gives the equivalent

integrated absorption. It is crucial to recognize that for the developments here and in the next section, the absorption parameters (such as line half width  $\alpha$  and intensity  $S$ ) are constant and independent of path). This is obviously unrealistic and we will discuss later how the results below can be modified to treat this added complexity.

(a) *Limits to the Integrated Absorption of a Single Line*

There are two extremely useful asymptotic limits of  $W(u)$  that occur repeatedly in discussion of molecular absorption.

- The weak line limit (linear limit)

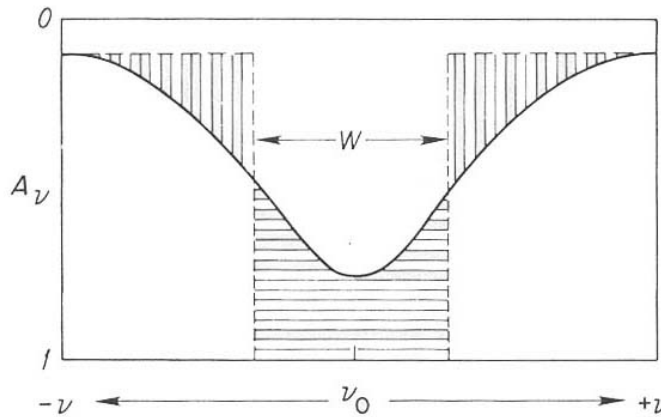


Fig. 9.2 Schematic interpretation of the equivalent width.

Suppose  $u \rightarrow 0$ ,  $Sf(\nu)u \ll 1$ , then

$$e^{-Sf(\nu)u} \sim 1 - Sf(\nu)u \tag{9.9b}$$

and

$$W = Su \int f(\nu) d\nu = Su \tag{9.10}$$

which is valid no matter what the line shape.

- Strong line or square root limit:

For this limit we consider the Lorentz line shape

$$f(\nu - \nu_0) = \frac{\alpha_L / \pi}{(\nu - \nu_0)^2 + \alpha_L^2}$$

Suppose  $|\nu - \nu_0| \gg \alpha_L$  so that  $\frac{1}{(\nu - \nu_0)^2 + \alpha_L^2} \rightarrow \frac{1}{(\nu - \nu_0)^2}$  then

$$\exp\left[-\frac{Su\alpha_L/\pi}{(v-v_o)^2+\alpha_L^2}\right] \rightarrow e^{-\frac{Su\alpha_L}{\pi(v-v_o)^2}}$$

and

$$W(u) = \int dv \left[ 1 - e^{-\frac{Su\alpha_L}{\pi(v-v_o)^2}} \right]$$

and

$$W(u) = 2\sqrt{Su\alpha_L}. \quad (9.11)$$

Conditions of strong absorption occur either as a result of abundant  $u$  absorber and/or high pressure (i.e., large  $\alpha_L$ ). Weak absorption is sensitive to abundance. A physical interpretation of these limits is afforded by reference to Fig. 9.3. In the linear region absorption occurs at the center of the line. A point is reached where all the energy is removed from the line center so that as  $u$  increases, the absorption increases through the wings (strong region).

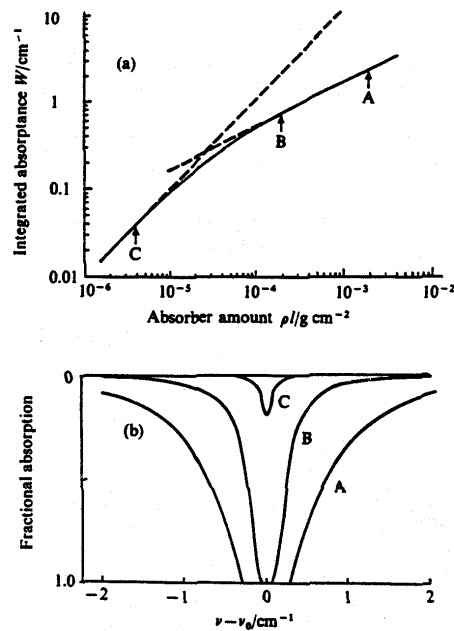


Fig. 9.3 The physical interpretation of strong and weak line absorption.

### (b) Broadband Absorption by a Single Lorenz Line

The equivalent width of a single Lorenzian line is expressed by the Ladenburg-Reiche function,

$$W_{L-R}(u) = \int_{-\infty}^{\infty} \left[ 1 - e^{-\frac{Su\alpha_L/\pi}{(v-v_o)^2+\alpha_L^2}} \right] dv = 2\pi\alpha_L \left[ \underbrace{ye^{-y}[I_0(y) + I_1(y)]}_{L-R \text{ function}} \right] \quad (9.11a)$$

where  $y = \frac{Su}{2\pi\alpha_L}$ . This can be usefully approximated by

$$W_{L-R} \sim Su \left[ 1 + \left( \frac{Su}{4\alpha_L} \right)^{5/4} \right]^{-2/5} \quad (9.11b)$$

within 1% for all values of  $Su/\alpha_L$ . Figure 9.3 (upper panel) provides a schematic demonstration of the strong and weak absorption and the  $L-R$  function. We call such plots 'the curve of growth' and these are fundamental to very important topics of atmospheric radiation.

*(b) Absorption by Lines with Distributed Line Intensities*

Here we consider the absorption averaged over lines that vary in intensities from line to line but not in their width and do not overlap in any way. Since the variation in line intensity over a band of thousands of lines is much more significant than is the variation of  $\alpha_L$ , this is a reasonable approximation. Furthermore, we will see how certain models of bands of overlapping lines reduce to this simple distribution of single lines.

Figure 9.4 is a diagram of the line intensity distribution expressed as the function  $p(S)$  where  $p(S)dS$  is the fraction of lines having intensities between  $S$  and  $S + dS$ . There have been different models derived according to the assumed form of  $p(S)dS$  and we will now consider two specific examples:

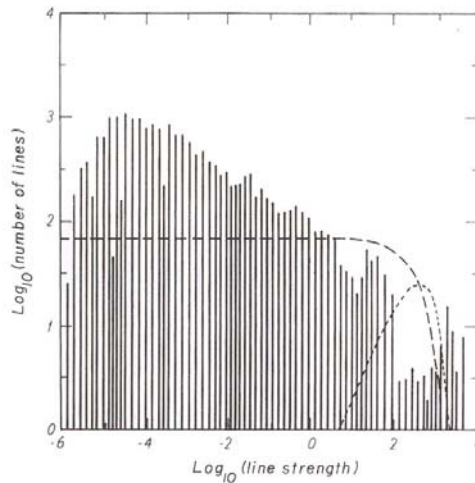


Fig. 9.4 A histogram (vertical bars) of all  $CO_2$  lines at wave numbers between  $450$  and  $900 \text{ cm}^{-1}$ . Each bar represents the number of lines in a given line group. Lines with strengths within 20% of the mean strength of a given group are gathered into that group. Analytic line-strength distributions obtained with the Goody (dotted line) and Malkmus (long-dashed line) models are also shown (after Crisp et al., 1986, with modifications).

- Goody (1952)

$$P(S) = \frac{1}{\sigma} \exp \left[ -\frac{S}{\sigma} \right]$$



- Malkmus (1967)

$$P(S) = \frac{1}{S} \exp\left[-\frac{S}{\sigma}\right]$$

where  $\sigma$  is the mean line intensity,

$$\sigma = \int_0^{\infty} Sp(S)dS .$$

if

$$k_\nu = f(\nu)S$$

then it follows that

$$\bar{W} = \int_0^{\infty} p(S)W(S)dS$$

and

$$\bar{W} = \int_0^{\infty} p(S)dS \int_{-\infty}^{+\infty} 1 - \exp[-Sf(\nu)u]d\nu$$

Thus:

- Goody

$$W = \int_{-\infty}^{\infty} \sigma f(\nu)u / [1 + Sf(\nu)u]d\nu$$

$$W_G = \sigma u \left[ 1 + \frac{Su}{\pi\alpha_L} \right]^{-1/2} \tag{9.12a}$$

- Malkmus

$$W = \int_{-\infty}^{\infty} \ln[1 + uSf(\nu)]d\nu$$

$$W_M = \frac{\pi\alpha_L}{2} \left[ \left( 1 + \frac{4Su}{\pi\alpha_L} \right)^{1/2} - 1 \right] \tag{9.12b}$$

## 9.4 Overlapping Lines: Band Models

It is obvious that over some interval  $\Delta\nu$ , increasing the optical mass ( $u$ ) cannot yield an increase in absorptance indefinitely if several overlapping lines are present in  $\Delta\nu$ . Thus the square root formula must fail. Attempts have been made to modify single line absorption theory to include line overlap—but these

on the whole are not fruitful. More successful are the approaches adopted based on treating the array of lines as a statistical entity rather than as a group of individual lines. Models of this type are referred to as *statistical band models*.

(a) *Regular Model*

Elsasser, 1938: Mean absorption and equivalent absorption coefficient of a band spectrum. *Phys. Rev.*, **54**, 126-129. Goody and Yung, 4.5. This model is most closely met for  $P + R$  branches of linear molecules.

$$f_E(\nu) = \frac{1}{\delta} \frac{\sin h(2\pi\alpha/\delta)}{\cosh(2\pi\alpha/\delta) - \cos(2\pi\nu/\delta)} = \sum_{n=-\infty}^{n=+\infty} \frac{1}{\pi} \frac{\alpha_L}{(\nu - n\delta)^2 + \alpha_L^2} \quad (9.13)$$

The corresponding transmission for homogeneous paths is

$$\bar{T}_E = \frac{1}{\Delta\nu} \int_{\Delta\nu} \exp[-Sf_E(\nu)u] d\nu$$

where  $\Delta\nu = 6$ . This integral cannot be solved in terms of elementary functions.

Consider two limits

- $\frac{\alpha_L}{\delta} \rightarrow \infty$ ,  $\sin h2\pi/\delta$ ,  $\cos h2\pi\alpha/\delta \rightarrow \infty$  and

$$T_E = \exp[-Su/\delta]$$

Here lines strongly overlap and there is no line structure. Further increase of  $\alpha_L/\delta$  (i.e., pressure) has no effect on the continuum. Transmission is independent of line shape.

- $\sin h2\pi\alpha_L \sim 2\pi\alpha_L/\delta$ ,  $\cos 2\pi\alpha_L/\delta \sim 1$  (small  $\alpha_L/S$ )

$$\bar{T}_E = 1 - \Phi\left(\sqrt{\pi S \alpha_L u / \delta}\right) = 1 - \Phi\left(\sqrt{\frac{\ell u}{2}}\right), \ell = \frac{2\pi\alpha S}{\delta^2} \quad (9.14)$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx = \text{probability integral}$$

The agreement with observation for this model is excellent when applied to an appropriate absorber (Fig. 9.5).

(b) *Random Models (G + Y, p. 158)*

Whereas the application of a regular band model to molecular absorption in the atmosphere has limited scope, use of random band models has been far more widely used and validated against

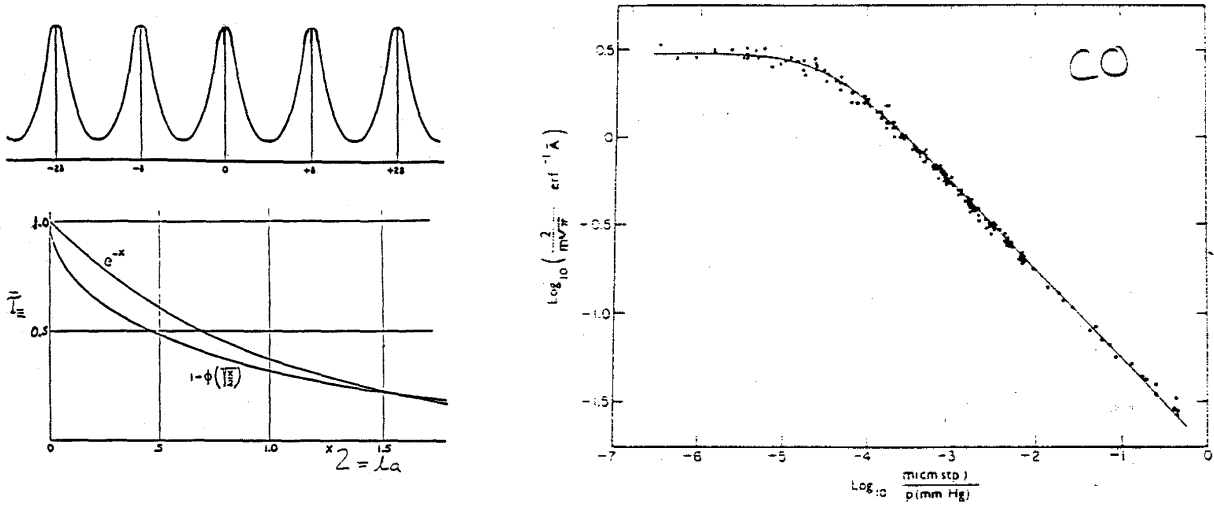


Fig. 9.5 (a) Line shape for the Elsasser model (after Goody, 1964). (b) Comparison of transmission for purely exponential type (single line) and the Elsasser band model of regularly overlapping lines, (c) Measured and fitted transmissions for a CO absorption band.

observations. One approach to the development of a random model is to take an infinite array (like the Elsasser model) and then combine a number of these arrays by multiplication. Consider for illustration a band of constant line intensity, then

$$k_\nu = \sum_{i=1}^N k_\nu^{(i)}$$

is the absorption coefficient at  $\nu$  due to the superposition of  $N$  lines distributed randomly in the interval  $-N\delta/2$  and  $N\delta/2$  by lines located at  $\nu_i$ . The transmission is

$$T_\nu = \exp\left(-u \sum_{i=1}^N k_\nu^{(i)}\right) = \prod_{i=1}^N \exp(-uk_\nu^{(i)}) \quad (9.15)$$

If the probability that a single line lies in the interval  $d\nu_i$  is  $d\nu_i/\delta$ , then the joint probability that there are lines between  $\nu_1$  and  $\nu_1 + d\nu_1$ ,  $\nu_2$  and  $\nu_2 + d\nu_2$ , and so on is

$$\prod_{i=1}^N \frac{d\nu_i}{\delta} \quad (9.16)$$

For all possible arrangements of lines in the interval

$$\bar{T} = \frac{\prod_{i=1}^N \int_{-N\delta/2}^{N\delta/2} \frac{d\nu_i}{\delta} \exp[-uk^{(i)}]}{\prod_{i=1}^N \int_{-N\delta/2}^{N\delta/2} d\nu_i / \delta} \quad (9.17)$$

with some approximation ( $n \rightarrow \infty$ )

$$\begin{aligned}\bar{T} &\rightarrow \exp\left[-\frac{1}{\delta} \int_{-\infty}^{\infty} 1 - \exp[-uk_v] dv\right] \\ &= \exp[-W/\delta]\end{aligned}\quad (9.18)$$

This states that the transmission through a random array of lines equals the exponential of the mean absorption ( $W/\delta$ )

Consider now  $M$  such arrays of random lines superimposed on one spectral interval  $M\delta$  wide, then

$$\bar{T}_i = \exp[-W_i/M\delta] \quad (9.19)$$

where  $W_i$  is the equivalent width of one line in the ' $i^{\text{th}}$ ' array. Since transmission is exponential,

$$\bar{T} = \prod_{i=1}^M \bar{T}_i = \exp\left[-\frac{1}{M\delta} \sum_i W_i\right] = \exp[-\bar{W}/\delta] \quad (9.20)$$

Now the average absorption  $\bar{W}$  was derived according to Eqns. (9.12a) and (9.12b) for Goody and Malkmus line intensity distribution. Thus:

$$\begin{aligned}\bar{T}_{\text{Goody}} &= \exp\left[\frac{-\bar{S}u/\delta}{(1 + \bar{S}u/\pi\alpha_L)^{1/2}}\right] \\ \bar{T}_{\text{Malkmus}} &= \exp\left[\frac{-\pi\alpha_L}{2\delta} \left[\left(1 + \frac{4\bar{S}u}{\pi\alpha_L}\right)^{1/2} - 1\right]\right]\end{aligned}\quad (9.21)$$

Clearly lines are not randomly distributed (they are predicted by quantum mechanical formula) and so a random band model is just an approximation to the actual transmission by a band of overlapping lines. The viability of the model, however, can be tested against laboratory data—with very good agreement (Fig. 9.6).

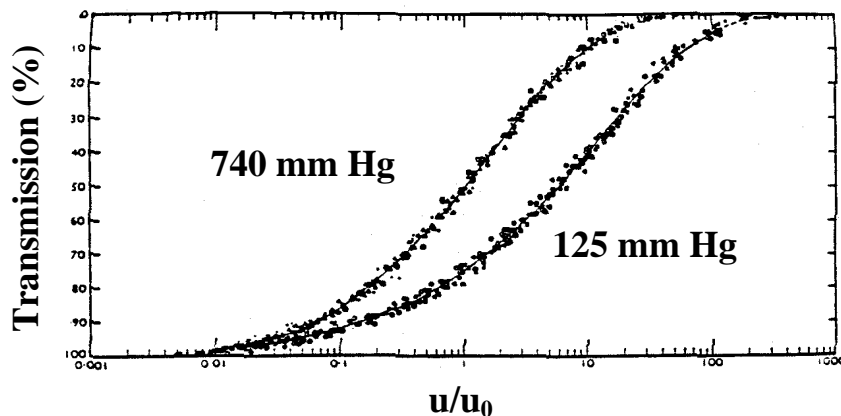


Fig. 9.6 Comparison between the random model (full line and observation (points) for sections of the  $6.3 \mu\text{m}$ ,  $2.7 \mu\text{m}$ ,  $1.87 \mu\text{m}$ ,  $1.38 \mu\text{m}$ , and  $1.1 \mu\text{m}$  band of water vapor. The different symbols represent absorptions by different bands.

(c) *Band Parameter Fits (G + Y, p. 158)*

The idea of a band model is to use Eqn. (9.21) to fit actual spectroscopic data to deduce the band parameters, namely  $\alpha_L$ ,  $\delta$ ,  $\sigma$ . We will not discuss the actual methods by which band models are matched to observations to provide these parameters. It suffices to state that this is done by fitting in the strong and weak limits of absorption, using Eqns. (9.18a) and (9.18b) in the form

$$\mathcal{T}_{\text{Goody}} = \exp\left[-(w^{-2} + s^{-2})^{1/2}\right]$$

$$\mathcal{T}_{\text{Malkmus}} = \exp\left(-\frac{s^2}{2w}\left[\left(1 + \frac{4w^2}{s^2}\right)^{1/2} - 1\right]\right)$$

where we define the  $w$  (weak) and  $s$  (strong) parameters as

$$w = \frac{1}{\Delta\nu} \sum_i S_i u$$

$$s = \frac{1}{\Delta\nu} \sum_i \sqrt{S_i \alpha_i}$$

*Table 9.2*

Band	Interval (cm <sup>-1</sup> )	$\bar{S} / \delta$ (cm <sup>2</sup> g <sup>-1</sup> )	$\pi\alpha/\delta$
H <sub>2</sub> O rotational	40-160	7210.30	0.182
	160-280	6024.80	0.094
	280-380	1614.10	0.081
	380-500	139.03	0.080
	500-600	21.64	0.068
	600-720	2.919	0.060
	720-800	0.386	0.059
	800-900	0.0715	0.067
CO <sub>2</sub> 15 $\mu\text{m}$	582-752	718.7	0.448
O <sub>3</sub> 9.6 $\mu\text{m}$	1000.0-1006.5	$6.99 \times 10^2$	5.0
	1006.5-1013.0	$1.40 \times 10^2$	5.0
	1013.0-1019.5	$2.79 \times 10^3$	5.0
	1019.5-1026.0	$4.66 \times 10^3$	5.5
	1026.0-1032.5	$5.11 \times 10^3$	5.8
	1032.5-1039.0	$3.72 \times 10^3$	8.0
	1039.0-1045.5	$2.57 \times 10^3$	6.1
	1045.5-1052.0	$6.05 \times 10^3$	8.4
	1052.0-1058.5	$7.69 \times 10^3$	8.3
	1058.5-1065.0	$2.79 \times 10^3$	6.7
H <sub>2</sub> O 6.3 $\mu\text{m}$	1200-1350	12.65	0.089
	1350-1450	134.4	0.230
	1450-1550	632.9	0.320
	1550-1650	331.2	0.296
	1650-1750	434.1	0.452
	1750-1850	136.0	0.359
	1850-1950	35.65	0.165
	1950-2050	9.015	0.104
	2050-2200	1.529	0.116

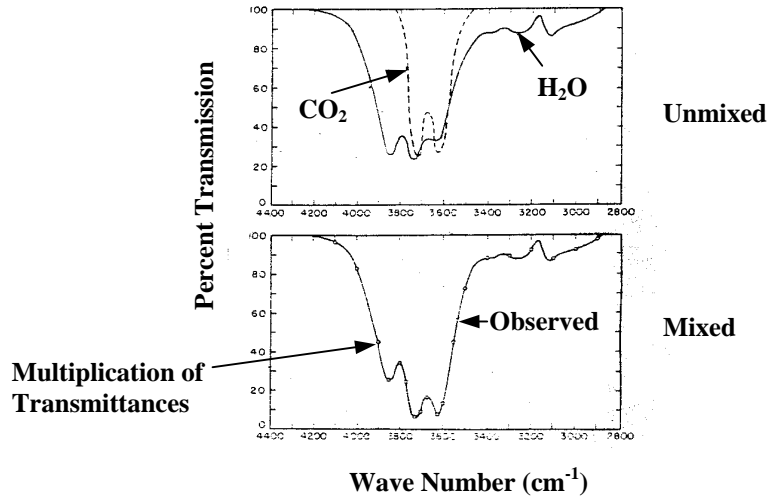


Fig. 9.7 Absorption in the spectral region from 4400-2800  $\text{cm}^{-1}$  where  $\text{CO}_2$  and  $\text{H}_2\text{O}$  overlap.

**Example 9.3:** Transmission in the  $\text{CO}_2$ - $\text{H}_2\text{O}$  overlap band. In the 15  $\mu\text{m}$  region, the transmission associated with two overlapped absorption bands has the form

$$T_{\text{H}_2\text{O}+\text{CO}_2} = T_{\text{H}_2\text{O}} \times T_{\text{CO}_2}$$

From the band parameters listed in Table 9.2, we have

$\text{CO}_2$	$s/\delta = 718.7$	$\pi\alpha/\delta = 0.448$
$\text{H}_2\text{O}$	$s/\delta = 2.919$	$\pi\alpha/\delta = 0.06$

then

$$T_{\text{H}_2\text{O}} = \exp \left[ \frac{-2.919 \times \beta u_{\text{H}_2\text{O}}}{\left(1 + \frac{2.919}{0.06} \times \beta u_{\text{H}_2\text{O}}\right)^{1/2}} \right]$$

$$T_{\text{CO}_2} = \exp \left[ \frac{-718.7 \times \beta u_{\text{CO}_2}}{\left(1 + \frac{718.7}{0.448} \times \beta u_{\text{CO}_2}\right)^{1/2}} \right]$$

where  $u_{\text{H}_2\text{O}}$  and  $u_{\text{CO}_2}$  are the respective path lengths of water vapor and carbon dioxide under consideration. A typical column value of water vapor (see Example 9.2) is  $u_{\text{H}_2\text{O}} = 2.8 \text{ gcm}^{-2}$  and a typical value of the column carbon dioxide path is  $u_{\text{CO}_2} = r_s p_s / g \approx 44 \times 330 \times 101300 / (980 \times 29) = 0.5 \text{ gcm}^{-2}$ . These values together with  $\beta = 1.66$  lead to

$$T_{\text{H}_2\text{O}+\text{CO}_2} = 0.406 \times 6.6 \times 10^{-8} = 2.68 \times 10^{-8}$$

and for double the amount of  $\text{CO}_2$ ,

$$T_{\text{H}_2\text{O}+\text{CO}_2} = 0.406 \times 7.0 \times 10^{-11} = 2.84 \times 10^{-11}$$

We conclude that the  $\text{CO}_2$  portion of the band is highly opaque and increases in this absorber only marginally reduce the already small transmission.

## 9.5 The Method of $k$ Distribution

The  $k$ -distribution method for transmission is based on grouping of the absorption coefficients  $k_\nu$  in some spectral interval (or band) (Fig. 9.8). In a homogeneous atmosphere, the spectral transmittance is independent of the ordering of  $k$  for a given spectral interval. Hence, the wave number integration may be replaced by an integration over the  $k$  space. If the normalized probability distribution function for  $k_\nu$  in the interval  $\Delta\nu$  is given by  $f(k)$  and its minimum and maximum values are  $k_{min}$  and  $k_{max}$ , respectively, then the spectral transmittance may be expressed by

$$\mathcal{T}(u) = \frac{1}{\Delta\nu} \int_{\Delta\nu} e^{-k_\nu u} d\nu = \int_0^\infty e^{-k_\nu u} f(k) dk$$

where  $k_{min} \rightarrow 0$  and  $k_{max} \rightarrow \infty$

$$\int_0^\infty f(k) dk = 1$$

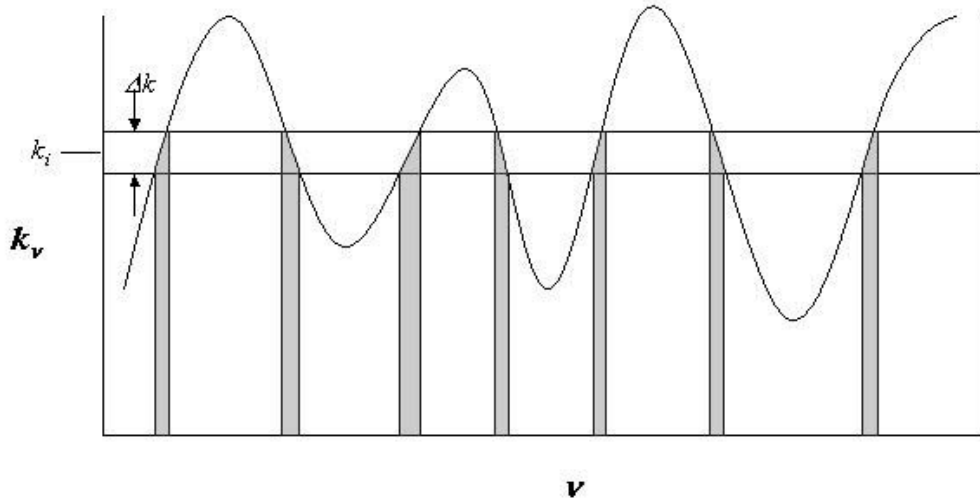


Fig. 9.8 The concept of the  $k$ -distribution approach. Divide the plot into  $n$  horizontal slices, centered on values  $k_1, k_2, \dots, k_n$ .  $F_i$  denotes the area of the  $\nu$  axis covered by points where  $k_i - \frac{\Delta k}{2} \leq k_\nu \leq k_i + \frac{\Delta k}{2}$ .

Moreover, a cumulative probability function may be defined in the form

$$g(k) = \int_0^k f(k) dk \quad (9.22)$$

where  $g(0) = 0$ ,  $g(k \rightarrow \infty) = 1$ , and  $dg(k) = f(k)dk$ . By definition,  $g(k)$  is a monotonically increasing and smooth function in  $k$  space. By using the  $g$  function, the spectral transmittance can be written

$$\mathcal{T}(u) = \int_0^1 e^{-k(g)u} dg = \sum_{j=1}^M e^{-k(g_j)u} \Delta g_j \quad (9.23)$$

Since  $g(k)$  is a smooth function in  $k$  space, the inverse will also be true here: that is,  $k(g)$  is a smooth function in  $g$  space. Consequently, the integration in  $g$  space, which replaces the tedious wave-number integration, can be evaluated by a finite and relatively small number of exponential terms.

The steps to implementing the  $k$ -distribution approach are highlighted schematically in Fig. 9.9. Figure 9.9(a) shows the spectrum of  $k_v$  in a portion of the  $9.6 \mu\text{m}$   $\text{O}_3$  band at a pressure of 30 mb and a temperature of 220 K. Figure 9.9(b) shows the probability distribution  $f(k)$  as a function of  $k$  derived from this spectrum (we will not discuss the details of how this is done although it is portrayed in Fig 9.8 and discussed further in G+Y). In Fig. 9.9(c) the cumulative probability function  $g(k)$  is shown as a function of  $k$ . We may then compute  $k(g)$  as a function of  $g$  from Eqn. (9.22). This curve is illustrated in Fig. 9.9(d). Since  $g$  is a smooth monotonic function, a few quadrature points suffice to achieve a high degree of accuracy in the transmittance computations.

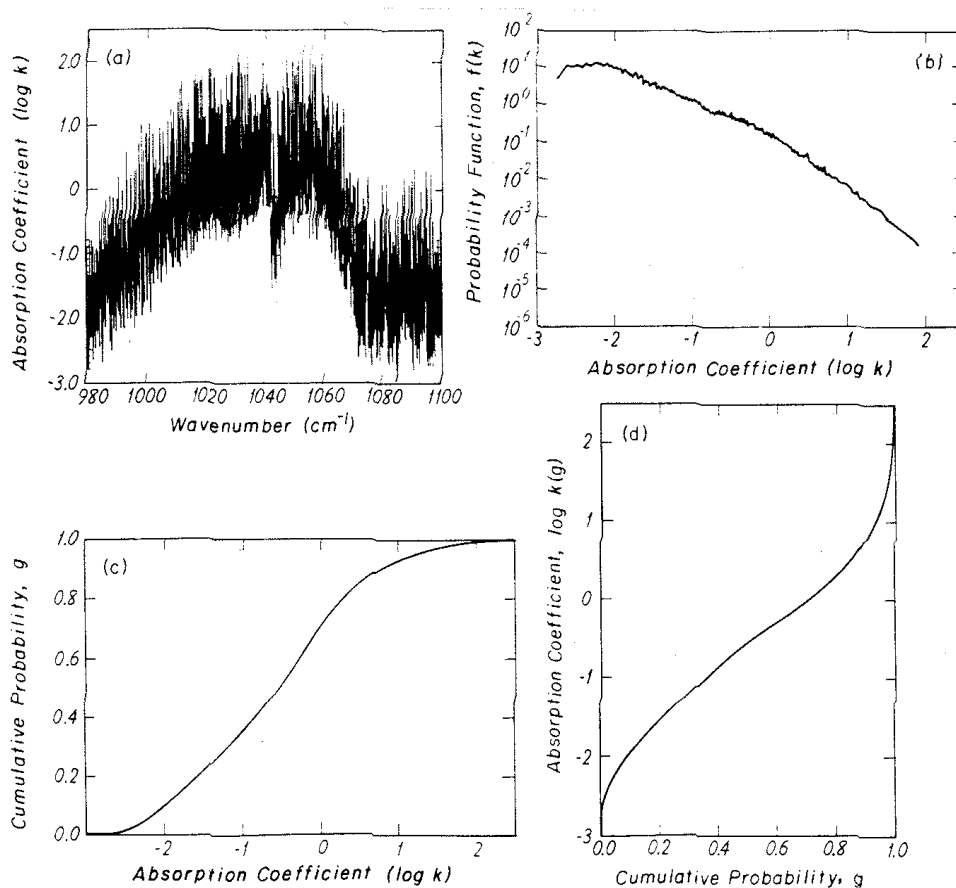


Fig. 9.9 (a) Absorption coefficient  $k_v$  in units of  $\text{cm}^{-1}\text{atm}^{-1}$  as a function of wave number with a resolution of  $0.05 \text{ cm}^{-1}$  in the  $9.6 \mu\text{m}$   $\text{O}_3$  band  $p = 30 \text{ mb}$  and  $T = 220 \text{ K}$ . (b) The probability function  $f(k)$  of the absorption coefficient. (c) The cumulative probability function of  $f(k)$  shown in (b), plotted as a function of  $k$ . (d) Same as (c), except that values of the absorption coefficient are expressed as a function of  $g$ .

The physical foundation for the  $k$  distribution is quite simple, but it offers clear advantages in the computation of broadband transmission. It has also been discussed by Domoto (1974) on some aspects of the theoretical foundation and the Laplace transforms for a number of band models. The idea of



scrambling and ranking absorption lines was described in the work of Ambartsumian (1936) of stellar atmospheres.

There is a second way to approach the  $k$  distribution and it follows from a closer look at Eqn. (9.22). It follows by definition that the transmission can be expressed as

$$\mathcal{T}(u) = \mathcal{L}[f(k)]$$

where  $\mathcal{L}$  is the Laplace transform. Thus the transmission is the Laplace transform of  $f(k)$  and this distribution is obtained as the inverse transform

$$f(k) = \mathcal{L}^{-1}[\mathcal{T}(u)]$$

For some functions, this provides a convenient way to obtain the spectral function  $f(k)$ . As it turns out, the inverse Laplace transform of the Malkmus model is obtained analytically as

$$kf(k) = \frac{1}{2} \left( \frac{\bar{k}y}{k} \right)^{1.2} \exp \left[ \frac{\pi y}{4} \left( 2 - \frac{k}{\bar{k}} - \frac{\bar{k}}{k} \right) \right]$$

where  $\bar{k} = \sigma / \delta$  and  $y = \alpha_L / \delta$ .

## 9.6 Selected Empirical Transmission Functions

A wide variety of empirical transmission models based on laboratory measurements have been employed in the literature. For example:

$$\begin{aligned} W_{LAB} &= A\tilde{u} & W < W_o \\ &= B + C \log \tilde{u} & W > W_o \\ \tilde{u} &= \tilde{u}(p / p_{LAB})^n \end{aligned}$$

where  $A$ ,  $B$ ,  $C$ ,  $W_o$ , and  $n$  are empirical constants.  $\tilde{u}$  is absorber mass,  $p$  is pressure (subscript "LAB" refers to laboratory conditions). Most only involve single path parameter  $u$ . All empirical models should be used with caution: unless based on theory, applicable only to a range of parameters for which they are fitted.

Two widely used empirical approximations to the solar weighted broadband absorption function

$$\bar{A}(u) = \frac{\int F_{\odot, \lambda} A_{\lambda}(u) d\lambda}{Q_{\odot}}$$

for the path  $u$  were derived by Lacis and Hansen (1974) for UV and visible ozone absorption and near infrared water vapor absorption. (The transmission is just  $1 - \bar{A}$ ). The formulae are:

$$A_{oz}^{vis}(\Omega) = \frac{0.02118X}{1 + 0.042X + 0.000323X^2} \quad (9.24a)$$

$$A_{oz}^{uv}(\Omega) = \frac{1.082X}{(1 + 138.6X)^{0.805}} + \frac{0.0658X}{1 + (103.6X)^3} \quad (9.24b)$$

where  $X$  is the ozone amount for the slant path expressed in cm STP and  $X = \Omega m_r$ , where  $\Omega$  is the column ozone amount above some specified level

$$m_r = \frac{35\mu_o}{1224\mu_o^2 + 1}$$

and  $\mu_o = \cos \theta_o$ . This factor (referred to as the relative airmass) differs only from  $\sec \theta_o$  for  $\theta_o$  near  $90^\circ$  due to refraction effects of the solar beam at these glancing angles.

The formula for water vapor is

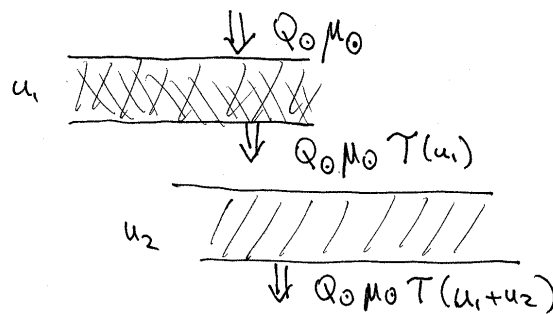
$$A_{wv}(w) = \frac{2.9u}{[(1 + 141.5u)^{0.635} + 5.925u]} \quad (9.24c)$$

where  $u = wm_r$ , such that  $w$  is the column water vapor amount (precipitable water) in units of  $\text{gcm}^{-2}$  (this is equivalent to cm STP). The total broadband absorption with respect to the entire solar spectrum is

$$A \approx A_{wv}(u) + A_{oz}^{uv}(\Omega) + A_{oz}^{vis}(\Omega)$$

given that the three absorptions occur in three different portions of the solar spectrum such that they overlap in a simple additive way. Figure 9.10a presents comparisons of the two formulae against actual spectrally integrated ozone absorptions. Figure 9.10b shows the broadband water vapor absorption derived from a number of different sources of both absorption data and spectral solar flux. Much of the difference can be explained by the actual choice of  $F_{\odot\lambda}$  for integrating the spectral absorption. These differences lead to significant differences in calculating the solar flux.

**Example 9.4:** Broadband transmission of the direct solar beam. Consider the following



Two overlying absorbing layers of path  $u_1$  and  $u_2$ . In the shaded upper layer, the absorbed flux is

$$\Delta F_1 = Q_{\odot} \mu_{\odot} \bar{A}(u_1 m_r)$$

where  $m_r \approx 1/\cos \theta_o$ . The absorption in the lower layer is then

$$\Delta F_2 = Q_{\odot} \mu_{\odot} [\bar{A}((u_1 + u_2) m_r) - \bar{A}(u_1 m_r)]$$

The transmission through the two layers is  $T = 1 - \bar{A}((u_1 + u_2) m_r)$

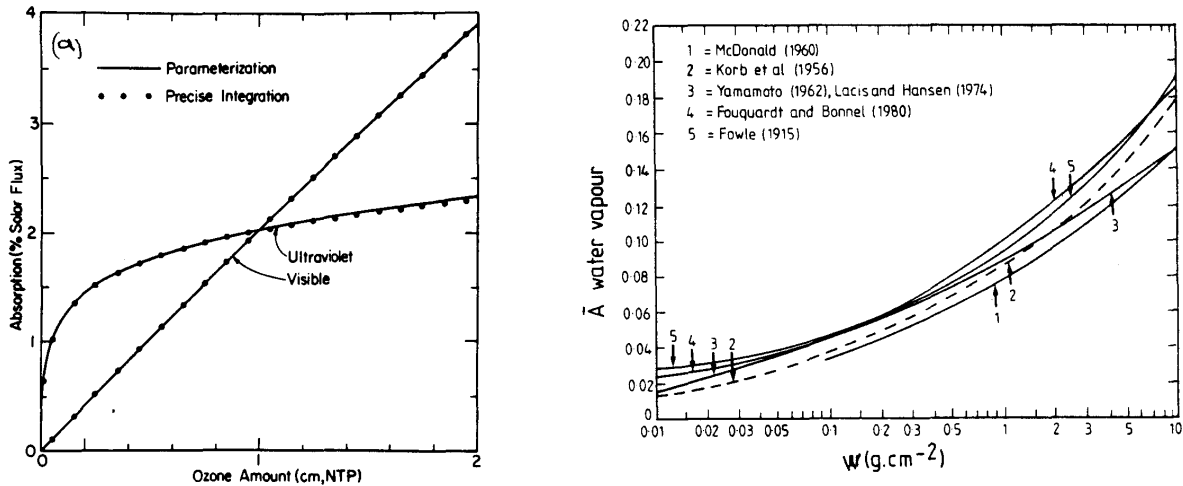


Fig. 9.10 (a) Percentage of the total solar flux absorbed as a function of ozone amount  $\Omega$ . (b) Same as (a) but as a function of water vapor amount.

### 9.7 Transmission Along Inhomogeneous Paths (Section 6.4 Goody and Yung)

So far all our discussion of transmission applies to the case of homogeneous paths (i.e., paths over which temperature and pressure and hence  $k(\nu)$  are constant) such as might arise in the laboratory. We now must modify this view as

- Most problems of transmission in the atmosphere apply to paths for which  $p$  and  $T$  vary.
- Laboratory data are obtained for fixed  $p$  and  $T$ , which might not be representative of atmospheric conditions and some adjustment is needed.

Figure 9.11 provides a schematic illustration of the consequence of transmission along a pressure varying path. The atmospheric line profile is no longer Lorentzian.

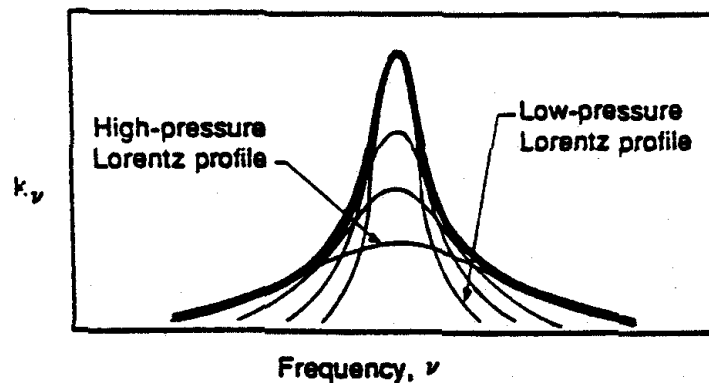


Fig. 9.11 Schematic composite showing how an actual line profile over a variable pressure path forms as a composite of the individual Lorentz profiles. The atmospheric line profile is not Lorentz in general: it is more sharply peaked because of low-pressure contributions, with broader wings due to high-pressure contributions.

In treating inhomogeneous effects, the assumption is made that the absorption for paths along which  $p$  and  $T$  vary can be approximated by absorption expressed in terms of a homogeneous path with the parameters scaled in some way. Two principal forms of scaling are used. Before discussing these, it is worthwhile considering one case for which an analytic solution exists.

(a) *Constant Mixing Ratio, Isothermal Atmosphere—An Exact Solution*

There is one hypothetical case for which the algebra can be done. Consider a line centered at  $\nu_o = 0$  for convenience, then  $\tau(\nu)$  has the form

$$\tau(\nu) = \int_{u_1}^{u_2} \frac{S(T)\alpha_o(p/p_o)}{\pi\nu^2 + [\alpha_o(p/p_o)]^2} du \quad (9.25)$$

$$du = \frac{r}{g} dp = m dp \quad (m = \text{absorber 'mass'})$$

Also, assume the property

$$rS = \text{constant} \quad (9.26)$$

such as occurs for an isothermal atmosphere ( $S$  constant) with a uniformly mixed absorber ( $r$  constant). Then

$$\tau(\nu) = \frac{Sr p_s}{\pi\alpha_o g} \int_{p_2}^{p_1} \frac{\tilde{p}}{(\nu/\alpha_o)^2 + \tilde{p}^2} d\tilde{p} \quad (9.27)$$

where  $\tilde{p} = p/p_o$  and

$$\tau(\nu) = \eta \left[ \log_e \left\{ \left( \frac{\nu}{\alpha_o} \right)^2 + p^{-2} \right\} \right]_{\tilde{p}_2}^{\tilde{p}_1} \quad \eta = Su / 2\pi\alpha_o \quad (9.28)$$

$$T_\nu = e^{-\tau(\nu)} = \left( \frac{\nu^2 + \alpha_1^2}{\nu^2 + \alpha_2^2} \right)^{-\eta}$$

for  $\alpha_i = \alpha_o \tilde{p}_i$ . Figure 9.12 in Example 9.5, shows the comparison between the transmission derived according to Eqn. (9.28) with  $-\eta = 1$  and the transmission calculated assuming the mean pressure  $\tilde{p} = (p_1 p_2)^{1/2}$  in the homogeneous path formula

$$\tau_\nu = \frac{S}{\pi} \frac{\alpha_o (\tilde{p}/p_o) u}{\nu^2 + [\alpha_o (\tilde{p}/p_o)]^2}$$

(b) *Scaling Approximation*

The simplest and most common way of dealing with nonhomogeneous paths is the 'scaling' or one parameter approximation. Let us start with the assumption that pressure and temperature effects on the absorption are separable according to

$$ku(p, T) = \Psi(\nu)\Phi(p)\chi(T) \quad (9.29)$$

Example of factorization:  $\nu - \nu_o > \alpha_L$  as occurs in line wings, then

$$\begin{aligned} k_\nu &= \frac{S\alpha_L/\pi}{(\nu - \nu_o)^2} \\ \Rightarrow \Psi(\nu) &\sim \left(\frac{1}{\nu - \nu_o}\right)^2, \quad \Phi(p) \sim \frac{p}{p_o}, \text{ and} \\ \chi &\rightarrow \left(\frac{T_o}{T}\right)^{1/2} \end{aligned}$$

Then

$$\tau_\nu = \int_{u_1(p_1, T_1)}^{u_2(p_2, T_2)} k_\nu(p, T) du(p, T)$$

approximates to

$$\tau_\nu = \overbrace{\Psi(\nu)\Phi(p_o)\chi(T_o)}^{k_\nu(p_o, T_o)} \int_{u_1}^{u_2} \frac{\Phi(p)}{\Phi(p_o)} \frac{\chi(T)}{\chi(T_o)} du \approx k_\nu(p_o, T_o) \tilde{u} \quad (9.30)$$

where

$$\tilde{u} = \int \frac{\Phi(p)}{\Phi(p_o)} \frac{\chi(T)}{\chi(T_o)} du$$

It is generally assumed that

$$\Phi(p) \sim p^n$$

and

$$\chi(T) \sim (T)^{-m}$$

thus

$$\tilde{u} = \int \left(\frac{p}{p_o}\right)^n \left(\frac{T_o}{T}\right)^m du \quad (9.31)$$

Table 9.3 provides some often used values of  $n$  and  $m$  for various absorbing gases.

Table 9.3: Generally accepted values of  $n$  and  $m$  for various absorbing species.

Gas	Spectral Region	$n$	$m$
Water vapor		0.9-1	0.45
Carbon dioxide	Shortwave	1.75	11-8
Ozone		0	0
Water vapor	Longwave	0.5-0.9	0.45
Carbon dioxide		1.75	11-8
Ozone		0.4	0.2

It is generally assumed that there is no foundation for Eqn. (9.25) other than it seems to work. While this statement is generally true, we see in the strong absorption limit that the absorption coefficient actually factors in this way with  $n = 1$ .

(c) *Two Parameter Approximations: The Van de Hulst - Curtis - Godson (VCG) Approximation*

The previous method relies only on a scaling of the absorber amount to correct for path inhomogeneities. In general  $n$  varies depending on the absorption regime ( $n = 1$  strong,  $n = 0$  weak) and so is poorly defined in general. Obviously, a better and more sophisticated approach would be to employ two disposable parameters to simulate the absorption (e.g.,  $u$  and  $n$  in the scaling approximation). The most useful two-parameter method proposed is the Curtis-Godson approximation, which attempts to define a scaled absorber amount specified for a mean pressure. The approach was developed independently by Curtis (1952) and Godson (1954) and earlier by Van de Hulst (1945) in a rather intriguing article (unfortunately in French)—thus I prefer to call the approximation VCG. The aim of the VCG approximation is to provide such a fit of the transmission. To discuss this approximation, consider isothermal paths (for convenience only). The criteria adopted are to match the absorptions exactly in the strong and weak limits. To proceed, we start with

$$\begin{aligned}\bar{\mathcal{A}} &= \frac{W}{\Delta\nu} = \frac{1}{\Delta\nu} \int_{\Delta\nu} 1 - \exp\left[\int -k_\nu du\right] d\nu \\ &= \frac{1}{\Delta\nu} \int_{\Delta\nu} 1 - \exp\left[\int Sf(\nu) du\right] d\nu\end{aligned}\tag{9.32}$$

- Weak limit

We obtain the weak-line limit directly by considering the exponent in Eqn. (9.32) as it approaches zero

$$\tilde{\mathcal{T}} = \int e^{-\int Sf(\nu) du} d\nu \approx \frac{1}{\Delta\nu} \int \left[1 - \int Sf(\nu) du\right] d\nu \approx 1 - \int S du$$

Since  $\int f(\nu) d\nu = 1$  for regular band models,  $\alpha_i$  and  $S_i$  are constant over the interval chosen and thus the VCG approximation in this context states.

$$S\tilde{u} = \int S du \quad (\text{weak limit})$$

or

$$\tilde{u} = \int du \quad (9.33a)$$

- Strong limit

The strong-line limit follows in an analogous way to the derivation of the strong limit for homogeneous paths. For the inhomogeneous case, in the strong limit where  $|\nu - \nu_o| \gg \alpha_L$  then for a single line

$$\tilde{T} = \frac{1}{\Delta\nu} \int_{\Delta\nu} \exp\left[-\int \frac{S\alpha_L/\pi}{(\tilde{\nu})^2} du\right] d\tilde{\nu}$$

where  $\tilde{\nu} = \nu - \nu_o$ . If

$$x = \tilde{\nu} \left[ \int \frac{S\alpha_L du}{\pi} \right]^{-1/2}$$

then

$$\tilde{T} = \frac{1}{\Delta\nu} \left[ \int_u S\alpha_L / \pi du \right]^{1/2} \int_{\Delta x} \exp\left(-\frac{1}{x^2}\right) dx$$

and since  $\Delta\nu \gg \alpha_L$ , the integral limits are effectively infinite. Thus

$$\tilde{T} \sim 1 - 2 \left[ \int (S\alpha_L du) \right]^{1/2}$$

and by matching the equivalent homogeneous limit, we obtain

$$\begin{aligned} \tilde{\alpha}\tilde{u} &= \int \alpha du \quad (\text{strong limit}) \\ \tilde{p}\tilde{u} &= \int \tilde{p} d\tilde{u}. \end{aligned} \quad (9.33b)$$

**Example 9.5:** Band model example revisited. Suppose that the vertical distribution of absorber has the form  $r(p) = r_s \bar{p}^3$  where  $\bar{p} = p / p_s$ . Then under the VCG approximation,

$$\tilde{u} = \int du = \frac{r_s p_s}{g} \int_0^1 \bar{p}^3 d\bar{p} = \frac{r_s p_s}{4g}$$

and that

$$\tilde{p}\tilde{u} = \int p du = p_s \int_0^1 \bar{p} du$$

since  $du = rd p/g$ ,

$$\tilde{p}\tilde{u} = \frac{p_s^2 r_s}{g} \int_0^1 \bar{p}^4 d\bar{p} = \frac{r_s p_s^2}{5g}$$

and thus  $\tilde{p} = 0.8 p_s$ . Now this may be simply applied to either the Goody or Malkmus band models in the following way. Consider the Goody band model: for a uniform path

$$T_{\text{Goody}}(u) = \exp \left[ \frac{(-\sigma / \delta) u}{(1 + \sigma u / \pi \alpha_L)^{1/2}} \right]$$

where for a nonuniform path

$$T_{\text{Goody}}(\tilde{u}) = \exp \left[ \frac{(-\sigma / \delta) \tilde{u}}{(1 + \sigma \tilde{u} / \pi \alpha_{L,s} \tilde{p})^{1/2}} \right]$$

where  $\alpha_{L,s}$  is the band line half width defined at pressure  $p_s$ . Using the parameters from our earlier example, with  $\tilde{u} = 2.8 \text{ gcm}^{-2}$  then

$$T_{\text{Goody}}(\tilde{u} = 2.8) = \exp \left[ \frac{-2.919 \times 2.8}{(1 + 2.919 \times 2.8 / 0.06 \times 0.8)^{1/2}} \right] = 0.536$$

compared to the homogeneous path value of 0.498.



**Example 9.6:** A test of the VCG. The accuracy of this approach can be tested for the hypothetical case considered above. Consider the atmospheric layer as shown below, which extends between pressure  $p_1 \rightarrow p_2$ . Set

$$p_2 = fp_1$$

then according to Eqn. (9.28)

$$\tau_v^{exact}(1,2) = \eta \ln \left[ \frac{v^2 + \alpha_L(1)^2}{v^2 + \alpha_L(2)^2} \right]$$

which can be written in the form

$$\tau_v^{exact}(1,2) = \eta \ln \left[ \frac{(v/\bar{\alpha}_L)^2 + 1/f}{(v/\bar{\alpha}_L)^2 + f} \right]$$

where  $\bar{\alpha}_L$  is the mean half width defined as  $\bar{\alpha}_L[\alpha_L(1)\alpha_L(2)]^{1/2}$ . The VCG approximation expresses the optical thickness in the form

$$\tau_v^{VCG}(1,2) = \frac{S\tilde{u}}{2\pi\tilde{\alpha}_L} \frac{2}{(v/\tilde{\alpha}_L)^2 + 1}$$

where  $\tilde{u}$  and  $\tilde{\alpha}_L$  are defined by Eqns. (9.33a) and (9.33b). It is straightforward to show that

$$\tilde{u} = \frac{rp_1}{g} (1 = f)$$

$$\tilde{\alpha}_L = \left( \frac{1+f}{2} \right) \alpha_L(1)$$

for the case considered here. Thus the optical thickness of the layer predicted by the VCG approximation in terms of  $f$  and  $\bar{\alpha}_L$  is

$$\tau_v^{VCG}(1,2) = 2\eta \frac{(1-f)}{(1+f)} \frac{2}{(v/\bar{\alpha}_L)^2 + 1}$$

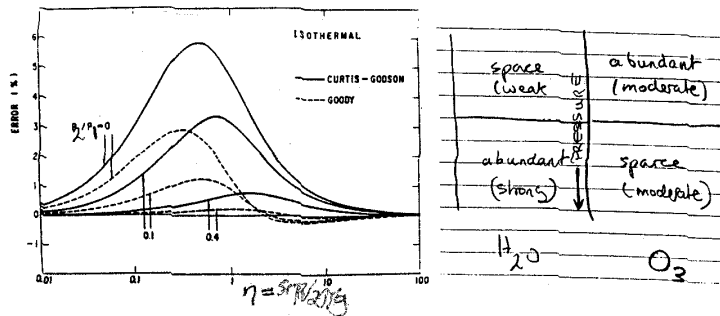


Fig. 9.12 Percentage error of the VCG for a single line and a constant mixing ratio for  $f = 0$ .

## 9.8 Problems

### Problem 9.1

Briefly explain or interpret the following:

- (a) Two sealed chambers contain the same amount of water vapor and are at the same temperature. One contains only water vapor, while the other holds a mixture of water vapor and air. Which has the smaller transmissivity averaged over a narrow spectral region containing a single water vapor absorption line?
- (b) The two sealed cells of (a) now both contain some amount of water vapor mixed in air. The concentration of water vapor in one cell is adjusted so that the transmission of 10  $\mu\text{m}$  radiation through one cell matches the transmission of 6.3  $\mu\text{m}$  radiation through the other cell. Which cell contains the most water vapor?
- (c) The temperature of both cells is now increased thus raising the pressure within the cell but assume no other changes occur. At which wavelength is the transmission a maximum (ignore any temperature effects on absorption)?

### Problem 9.2

Develop a relationship between the vertically integrated water vapor path through the entire vertical extent of the atmosphere (precipitable water) and the sea surface temperature. Assume

- (a) The vertical profile of specific humidity has the following form  $q_s(p/p_s)^\lambda$  where  $q_s$  is the surface specific humidity.
- (b)  $e_s \approx b \exp[a(T_s - T_o)]$ . Derive your answer in terms of the surface relative humidity,  $\lambda$ , and the SST  $T_s$ .

### Problem 9.3

Compute the optical path for:

- (a) Water vapor of a 100 mb thick homogeneous layer of mixing ratio  $r$ .
- (b) Total atmospheric  $\text{CO}_2$  if the mixing ratio is 330 ppm by volume.

### Problem 9.4

The following function

$$r(\psi) = r_p \frac{4a\psi^2}{(1 + a\psi^2)^2}$$

reasonably resembles the vertical profile of ozone mixing ratio such that with  $a = 1600$ , the maximum occurs at  $\psi = p/p_s = 0.025$ . Assuming a value  $r_p = 1 \times 10^{-5}$  kg/kg, derive the total column ozone and express your answer in Dobson units (the density of ozone at S.T.P. is  $2.14 \text{ kgm}^{-3}$ ).

### Problem 9.5

The rationale for the surface pressure measurement using two frequencies in the O<sub>2</sub> A band is discussed in Section 3.5. Given the definition of optical thickness, obtain an explicit form of the function  $t(p_s)$  given in (3.41) assuming (1) a Lorentz line and frequencies at the line center ( $\nu = \nu_0$ ), and (2) frequencies in the line wing  $|\nu - \nu_0| \gg \alpha_L$ . Neglect the effects of atmospheric temperature on line intensity and half-width. Express your answers in terms of  $S$ , the line strength;  $\alpha_0$  the line half width defined at some reference pressure  $p_0$  the mixing ratio  $r$  of the gas,  $p_{sat}$  the satellite pressure, and the acceleration by gravity  $g$ .

### Problem 9.6

Absorption in the atmospheric window between 8 and 13  $\mu\text{m}$  is represented by an absorption coefficient of the form  $k_2 e$  where  $e$  is the water vapor pressure (in kPa),  $k_2 \cong 10^{-1} (\text{g cm}^{-2})^{-1} \text{kPa}^{-1}$ . If the water vapor pressure near the surface is 1 kPa, calculate (1) the transmission of a horizontal path 1 km long near the surface, and (2) the transmission of a vertical path of atmosphere assuming that the distribution of water vapor pressure is proportional to pressure units of atmospheres raised to the fourth power.

### Problem 9.7

The absorption coefficient in the continuum has the form

$$k_\nu \approx k_{2,\nu} e$$

where  $e$  is the water vapor partial pressure in units of atmosphere. Assuming a hydrostatic atmosphere

$$p = p_s e^{-z/H}$$

where  $p_s = 1013.13 \text{ mb}$ , and assuming that the mixing ratio profile of water vapor is similarly exponential with

$$H_r = H/3$$

where  $H_r$  is the scale height of vapor

- (a) Derive an expression for the optical mass  $u$  for the vertical path from  $\tilde{p} = 0$  to  $\tilde{p}$  where  $\tilde{p} = p/p_s$  is the pressure in atmospheres. Express your answer in terms of  $r_s$ , the surface mixing ratio of water vapor, and  $\tilde{p}$ .
- (b) Assume that the temperature dependence of the absorption parameter  $k_{2,\nu}$  has the form

$$k_{2,\nu} = k_{2,\nu,s} / \tilde{p}$$

show that

$$T_\nu = \exp[-\beta \tilde{p}^7]$$

where

$$\beta = \frac{p_s r_s^2 k_{2,v,s}}{4.354g}$$

where  $e = r\tilde{p} / 0.622$ .

### Problem 9.8

Assume the following profile for water vapor mixing ratio:

$$r = r_s e^{-3z/H}$$

Calculate the broadband water vapor absorption of solar radiation in 10 adjacent 100 mb thick layers from the top of the atmosphere (0 mb) to 1000 mb and plot this absorption as a function of the mean layer pressure.

- (a) Contrast the vertical profiles of absorption assuming the following values of  $r_s$ : 5.4, 10.2 and 18.4  $\text{gkg}^{-1}$ .
- (b) Calculate the Planck weighted broadband flux absorption using a Goody band model and the parameters given in Table 4.4 (p. 11) for the rotation band and the vibration band. Calculate this transmission for a path extending through the column for the model atmosphere of 1 above (do only for  $r_s = 10.2 \text{ gkg}^{-1}$ ). Assume  $T = 270 \text{ K}$  in calculating the Planck Function.
- (b) Calculate the broadband transmission as in (2) above but for a path extending up from a reference level located at 800 mb to the top of the atmosphere and for a path extending downwards from this reference level to the surface. Plot these transmissions as a function of either pressure or altitude (your choice).