AT622 Section 2
Elementary Concepts of Radiometry

The object of this section is to introduce the student to two radiometric concepts—intensity (radiance) and flux (irradiance). These concepts are largely geometrical in nature. Neither quantity varies as light propagates along.

2.1 Frame of Reference

Before considering how we might describe electromagnetic wave propagating in space in radiometric terms, it is necessary to consider ways of representing the geometry of the flow. We use a terrestrially based frame of reference such as a Cartesian coordinate system and select one of its axes to be anchored in some way according to some property of the terrestrial atmosphere.

$\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors that define three orthogonal axes. Examples of two sun-based frames of reference, where the x axis points to the sun (i.e., the azimuth angle is defined relative to the sun's azimuth), are shown in Fig. 2.1. A general reference point within a Cartesian frame of reference may be indicated by the position vector $\vec{r}$ such that

$$\vec{r} = (x, y, z),$$

where (x, y, z) defines the coordinates of the tip of this vector.

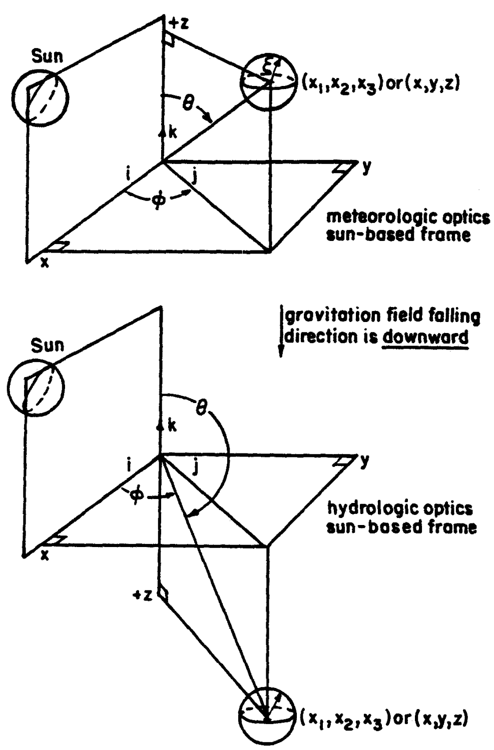

Fig. 2.1 Sun-based terrestrial frames of reference for meteorologic optics and hydrologic optics.

We define a direction vector in terms of a unit vector ($\vec{\xi}$) whose base is at the origin point and whose tip is the point (*a, b, c*) on the unit sphere that surrounds the origin. In this case, $\sqrt{a^2 + b^2 + c^2} = 1$. The unit direction vector may also be defined in terms of a general point (*x, y, z*) by

$$\vec{\xi} = \frac{\vec{r}}{|r|} = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{1/2}} .$$

A more trigonometrical interpretation of the direction vector follows by considering Fig. 2.2a. For a point (*a. b, c*) on the unit sphere, it follows that

$$\begin{aligned} a &= \vec{r} \cdot \vec{i} = \cos\phi \sin\theta \\ b &= \vec{r} \cdot \vec{j} = \sin\phi \sin\theta \\ c &= \vec{r} \cdot \vec{k} = \cos\theta = \mu . \end{aligned}$$

where θ is the zenith angle and ϕ is the azimuth angle. The latter, in this case, is measured positive counterclockwise from the *x* axis. Since $\vec{\xi}$ = (*a, b, c*), then

$$\vec{\xi} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) \tag{2.1}$$

are the three components of the direction vector. We will also use $\mu = \cos\theta$ throughout these class notes.

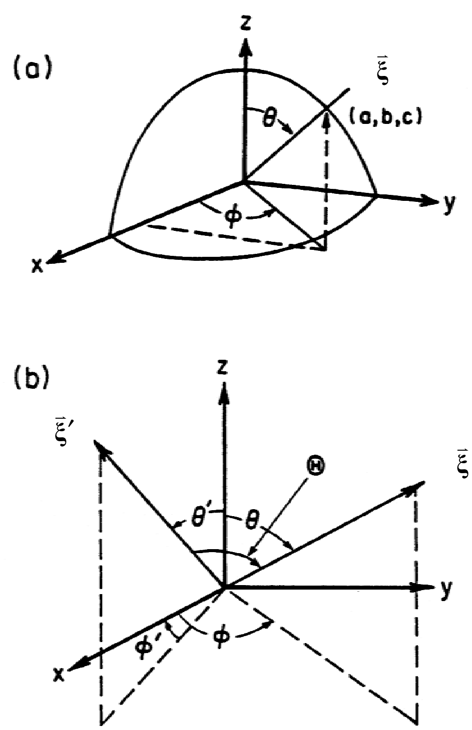

Fig. 2.2 (a) Angle and direction definitions defined with respect to a unit sphere. (b) Scattering geometry and the scattering angle on the unit sphere.

Example 2.1: Scattering angle

Many problems of interest require the definition of the angle formed between two directions. For example, the scattering angle Θ is the angle between the direction of incident radiation and the direction of the scattered radiation. If the former direction is $\vec{\xi}$ and the scattering direction is $\vec{\xi}'$ then

$$\cos\Theta = \vec{\xi} \cdot \vec{\xi}' .$$

We can schematically represent Θ and the two directions in question on a unit sphere (Fig. 2.2b). It follows from the above equation and Eqn. (2.1) that Θ can be stated in terms of two pairs of angles $\xi'(\mu', \phi')$ and $\xi(\mu, \phi)$

$$\cos\Theta = \mu\mu' + (1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\cos(\phi-\phi') .$$

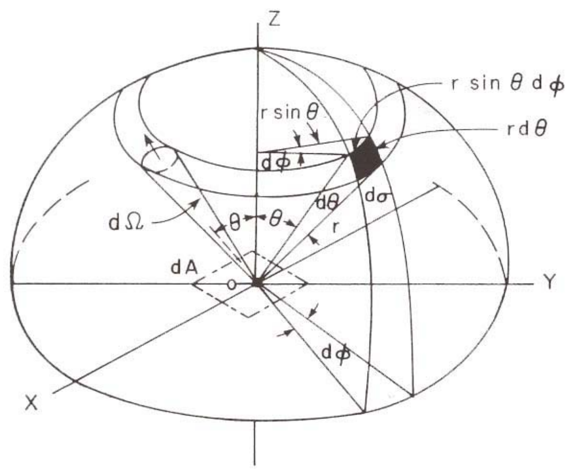

Fig. 2.3 Illustration of a solid angle and its representation in polar coordinates. Also shown is a pencil of radiation through an element of area dA in directions confined to an element of solid angle dΩ.

2.2 Solid Angle and Hemispheric Integrals

Many radiation problems, particularly those dealing with fluxes, require some type of integral over solid angle. A simple and convenient way to think about the solid angle is to imagine that a point source of light is located at the center of our unit sphere and that there exists a small hole of area A on its surface allowing light to flow through it. This light is contained in a small cone of directions, which is represented by the solid angle element Ω defined as

$$\Omega = \frac{A}{r^2} \tag{2.2}$$

With this definition, one can easily show that the solid angle with all directions around a sphere equals 4π. Referring to Fig. 2.3, one can write the differential area of the opening as

$$d\Omega = \sin\theta d\theta d\phi \,. \tag{2.3}$$

Integrating $d\Omega$ over the entire sphere

$$\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta = 4\pi \tag{2.4}$$

yields the result we intuited earlier in Eqn. (2.2).

Suppose we now wish to integrate some function, like the intensity, over a complete hemisphere of directions. To fix ideas, consider the intensity $I(\theta,\phi)$ flowing to some point on a horizontal surface from the hemisphere above it. The hemispheric integral of this intensity is then

$$h = \int_0^{2\pi} d\phi \int_{90°}^{0°} I(\theta,\phi)\sin\theta d\theta = \int_0^{2\pi} d\phi \int_0^1 I(\mu,\phi) d\mu$$

An even more important quantity in radiation studies is the hemispheric flux F defined as

$$F = 2\pi \int_0^1 I(\mu,\phi)\mu d\mu \,.$$

Note how this quantity differs from h above through the appearance of the factor μ in the integrand. The hemispheric flux defined in this way is a measure of the energy flowing through a horizontal surface per unit area and per unit time (we will discuss this in a more formal way later). Recall from Section 1 that the intensity is a measure of the energy flowing though a surface normal to the flow per unit area, per unit time, and per unit solid angle. The cosine factor therefore accounts for the projection onto a horizontal surface of the area that is normal to the flow of photons.

Example 2.2: Solid angle

1) The solid angle of a spherical segment is

$$\Omega(D) = \int_{\theta_1}^{\theta_2} \sin\theta d\theta \int_0^{2\pi} d\phi = 2\pi[\cos\theta_1 - \cos\theta_2] \,.$$

2) The solid angle of a spherical cap defined by the angle θ is

$$\Omega(D) = \int_0^{2\pi} d\phi \int_0^{\theta} \sin\theta d\theta = 2\pi[1-\cos\theta]$$

For small θ, $\cos\theta \to 1-\theta^2/2 + \ldots$ and

$$\Omega(D) \approx \pi\theta^2 \,.$$

3) The solid angle of the sun is therefore

$$\Omega_\odot = \pi\theta_\odot^2$$

whereas we shall see later, $\theta_\odot \approx r_\odot / R_{S_E}$, and

$$\Omega_\odot = \pi\left(\frac{0.7\times 10^6}{1.5\times 10^8}\right)^2 \approx 0.684\times 10^{-4} \quad \textit{steradian}$$

2.3 Basic Radiometric Concepts

Radiation is a way of transferring energy from one point to another and we now formalize a way of describing this flow. From Section 1 we learn how the energy of an EM wave is associated with the square of the amplitude of the *E* field. Now we consider the geometrical constructs of this flow of energy. These considerations are known as radiometry: **Radiation + Geometry.** Radiometry has almost become a discipline in itself—a large variety of terminologies and symbolisms exist. However, we need only consider one basic quantity from which others follow. Another point is that once we have established the nature of radiant energy, radiometry is by and large geometrical in nature.

The first basic quantity is the "radiant flux". The definition for "radiant flux" of monochromatic radiation is

$$P(v) = hv \times (n(v) \times c) \times dA \quad \mathrm{W(\mu m)^{-1}} \tag{2.5}$$

where $n(v)$ *is* the phase space density = number of photons per unit frequency per unit volume; hv *is* the energy of each photon; and $n(v)\times c$ is therefore the number of photons per unit frequency crossing a unit area per unit time. Two quantities that follow from *P* are:

$F_v = P(v)/A$ for the area density of radiant flux [$\mathrm{Wm^{-2}\ \mu m^{-1}}$]

which is strictly known as the flux density but we will call it flux (or irradiance and shortly dispense with the quantity *P*), and the monochromatic, or spectral intensity, or radiance

$I = P(v)/\Omega{\cdot}A$ [$\mathrm{W\ m^{-1} sr^{-1}\ \mu m^{-1}}$] (radiance)

Example 2.3: Photon flow rate of Example 1.2

Here we estimate the rate of photon flow required to deliver a given amount of flux at a specific wavelength. We begin with Eqn. (2.5)

$$F_v = \frac{P(v)}{dA} = n(v) \times c \times hv\,,$$

such that

$$n(v) \cdot c = \frac{F_v}{hv}$$

For $F = 0.1\ \mathrm{Wm^{-2}}$, $\lambda = 0.5\ \mu\mathrm{m}$, $c = 3 \times 10^8$ m/s, and $h = 6.6 \times 10^{-34}$ Js, one obtains

$$n(s) \cdot c = \frac{0.1 \times 0.5 \times 10^{-6}}{6.6 \times 10^{-34} \times 3 \times 10^{8}} = 2.5 \times 10^{17}$$

photons of $\lambda = 0.5\ \mu$m flow per sec through a unit area to produce 0.1 Watts of power per m^2.

There are only two ways that we need to visualize the flow of radiant energy (Fig. 2.4)—the first is as a function of direction and the second is as a function of space:

(1) For the first we imagine the radiant flux of energy *P(s)* passing through some area *dA* as in Fig. 2.4a.

(2) For the second depiction, radiant flux of energy *P(s)* passes through a single point *p* through a small set of directions $d\Omega$.

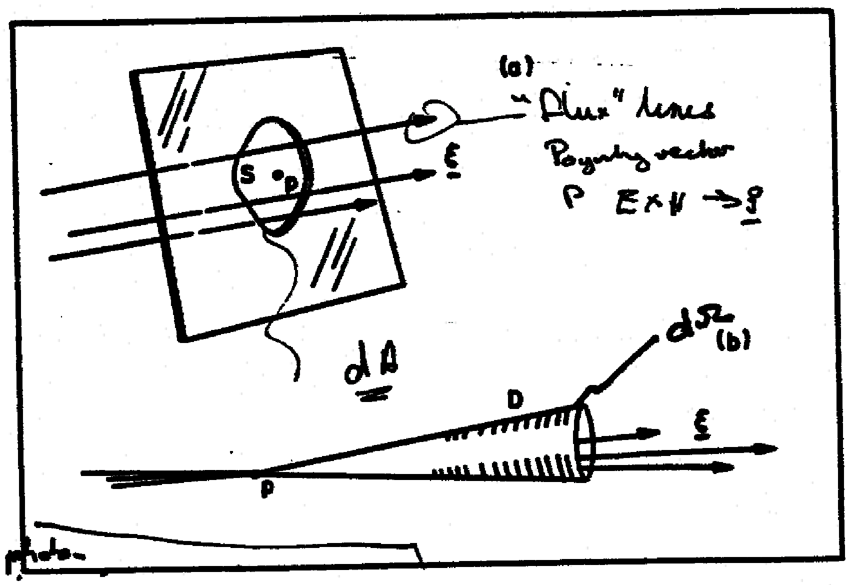

Fig. 2.4 (a) and (b) two hypothetical modes of flow of radiant energy.

(a) Spectral Intensity I

Unfortunately, the flows shown in 2.4a and b are not practical since these flow types are not measurable: an instrument detector can neither sense radiation at an infinitesimal point, since detectors have some characteristic area, nor can a detector measure purely parallel flow as they also have a characteristic angle. Consider a simple radiometer as shown in Fig. 2.5a. The detector subtends a solid angle

$$\Omega = a/\ell^2$$

where *a* is the area of the detector and ℓ is the length of the collimating tube. What is measured is then the quantity

$$I = \frac{P}{a \cdot d\Omega} \qquad [\mathrm{Wm^{-2}sr^{-1}\mu m^{-1}}] \tag{2.6}$$

which we will refer to as either spectral radiance or spectral intensity. Radiance (intensity) is a fundamentally important quantity as it is directly measurable by instruments we call radiometers. The product of the area and solid angle $a\Omega$ is known as the 'throughput' *T* of the radiometer and the radiance is then *P/T*.

(b) Irradiance or Flux F

An even more important radiometric quantity, at least from the perspective of meteorology and climatology, is the quantity referred to as irradiance (or flux density or just flux, again remember that we will shortly dispense with *P* for the rest of this class). This quantity describes the total flow of radiant energy that flows onto or from a surface. For a general definition of this quantity, consider a surface of area *dA* and flow from two directions $\bar{\xi}_1$ and $\bar{\xi}_2$, which make angles θ_1, and θ_2 with respect to the normal $\vec{n}$ to *dA*. The radiances I_1, and I_2 define the radiation field along each direction. The flux onto *dA* is then

$$f(\vec{n}) = \frac{P}{dA} = I_1(\bar{\xi}_1)(\bar{\xi}_1 \cdot \vec{n} d\Omega_1) + I_2(\bar{\xi}_2)(\bar{\xi}_2 \cdot \vec{n} d\Omega_2) \qquad \mathrm{Wm^{-2}}\mu\mathrm{m^{-1}}$$

and in the limit that the number of sources $\rightarrow \infty$, then

$$F(\vec{n}) = \int I(\bar{\xi}_i)\cos\theta d\Omega \qquad (2.7)$$

where *F* is defined with respect to the normal $\vec{n}$ **(Note: it is only meaningful to talk of flux relative to the orientation of some surface...most of our interests are for horizontal surfaces and thus $\vec{n}$ represents either the zenith or nadir).** When the integration is carried out over the entire sphere of solid angles, this is the *net* flux that flows through the surface. It is more common to carry out this integral in two parts, one over a positive hemisphere (positive in the sense that $\bar{\xi} \cdot \vec{n} > 0$, see Fig 2.5a) and one over the corresponding negative hemisphere (Fig. 2.5b).

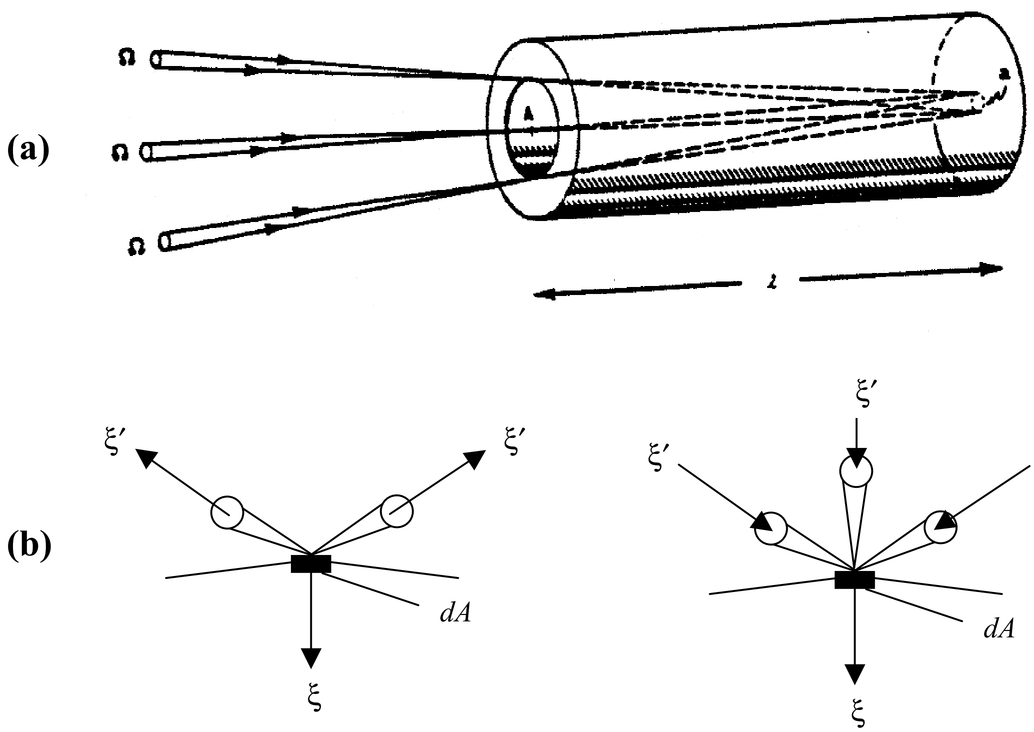

Fig. 2.5 (a) A simple radiometer and (b) hemisphere fluxes.

Example 2.4: Exploring the relation between radiance and flux

1) Consider the situation where radiation flows onto a surface defined by a discontinuity in refractive index. At the surface

$$\frac{P_1}{A_1} = F_1 = F_2 = \frac{P_2}{A_2}$$

Snell's law predicts that

$$m_1 \sin\theta_1 = m_2 \sin\theta_2 \qquad (\theta_1, \theta_2 \text{ are small by hypothesis})$$
$$m_1\theta_1 = m_2\theta_2$$

and it follows that

$$m_1^2\Omega_1 = m_2^2\Omega_2$$

where now we make use of our small cap approximation $\Omega = \pi\theta^2$. Since

$$\Omega_1 I_1 = F_1 = F_2 = \Omega_2 I_2$$

we obtain

$$\frac{I_1}{m_1^2} = \frac{I_2}{m_2^2}$$

Thus we take I/m^2 as the intensity when we are interested in propagation through an m varying media. The radiance from one m environment to another m environment thus needs to be adjusted by refractive index.

2) Hemispheric fluxes on a horizontal surface. The upward flux may be defined as

$$F^+ = \int_0^{2\pi} \int_0^{\pi/2} I(\theta,\phi)\cos\theta\sin\theta d\theta d\phi$$

and the downward flux is

$$F^- = \int_0^{2\pi} \int_{\pi/2}^{0} I(\theta,\phi)\cos\theta\sin\theta d\theta d\phi$$

and the net flux is $F = F^+ + F^-$. Often the limits of the θ integral for F^- are flipped, which in turn defines a positive F^- leading to an alternate definition $F = F^+ - F^-$. We will use this latter convention throughout. [Note also that the + sign on the upward flux means that the normal of the surface in question points upward along the vertical.]

Example 2.4: Continued.

3) Consider a special case of isotropic radiation (i.e., $I = I_0$ is constant), then it follows that

$$F^+ = \int_0^{2\pi} \int_0^{\pi/2} I_o \cos\theta \sin\theta d\theta d\phi = 2\pi I_o \int_0^1 \mu d\mu = \pi I_o$$

and also $F^- = \pi I_o$, *so* that $F^{net} = 0$. As much radiation flows onto the surface that leaves the surface.

4) Flux of an isotropic source on a vertical surface. Let us consider the surface in the y-z plane

$$\begin{aligned} F^{\pm} &= \int I(\xi)(\xi' \cdot \hat{i} d\Omega(\xi')) \\ &= I_o \int_{-\pi/2}^{\pi/2} \int_{\pi}^{0} \cos\theta \sin\theta d\theta d\phi = \pi I_o \end{aligned}$$

where we make use of Eqn. (2.1). Also take special note of the limits of the integration and the hemisphere these limits define.

5) The intensity and flux from the sun. We will see later how the sun radiates approximately as a blackbody of temperature $T_\odot$ = 5790 K. This radiation is emitted isotropically from the sun with a broadband intensity (i.e., at an intensity that has been integrated over all wavelengths)

$$I_\odot = \frac{\sigma}{\pi} T_\odot^4 = 2 \times 10^7 \qquad [\mathrm{Wm^{-2}sr^{-1}}]$$

If we consider the geometry as shown, then the flux from the sun incident on a surface whose normal is along the direction from the point P on the earth's surface to the center of the sun is

$$\begin{aligned} F_\odot &= \int_{\Omega_\odot} I_\odot \cos\theta d\Omega \\ &= I_\odot \int_0^{2\pi} d\phi \int_0^{\theta_c} \sin\theta \cos\theta d\theta \\ &= I_\odot \pi [\sin^2 \theta_c] = I_\odot \pi \frac{r_\odot^2}{R^2} = I_\odot \Omega_\odot \\ &\approx 1380 \qquad [\mathrm{Wm^{-2}}] \end{aligned}$$

Example 2.5: The black of night: Olbers' paradox

An ancient astronomer, if asked why the night sky is black, probably would have answered that it was because the sun is absent. If we then ask why the stars don't take the place of the sun, then the likely answer is because the stars are of limited number and individually dim. This last argument has lost its force over the centuries and astronomers tell us that the number of stars occupying the night sky is tremendous indeed. We are left with a paradox of sorts—why is not the night sky as brilliant as the daytime sky filled with the light from an almost infinite number of stars. Olbers pondered this paradox and approached it with the following assumptions:

1. The universe is infinite in extent,
2. The stars are infinite in number, and
3. The stars are of uniform average brightness through all space.

He then considered space as divided into concentric shells about the observer that are large enough to be populated by stars. The amount of light that reaches us from each star (think of this as the product of $I_{\odot}\Omega_{\odot}$) varies inversely as the square of its distance from us. But as we look farther out in space the volume of the shell of space also expands (as the distance squared) in such a way that the increased number of stars in the farther shell cancels with the decreased brightness of these more distant stars.

Thus the crux of the paradox is—if the universe is infinite in extent and thus consists of an infinite number of shells, the stars of the universe, however dim they may individually be, ought to deliver an infinite amount of light to Earth. Somewhere in Olbers' paradox there is some mitigating circumstance or logical error. It is commonly thought that the failure of the above arguments occurs with assumption (3). We know that the stars of distant galaxies are receding and this movement caused a red-shift in the spectrum. With the expansion of the universe each succeeding shell delivers less light as it is subject to a successively greater red shift. Thus we receive only a finite amount of energy from the universe and the night sky is black.

2.4 Problems

Problem 2.1

Radiance of the moon and sun

(a) Calculate the solid angle subtended by the sun, and the solid angle subtended by the moon as seen from the center of the Earth.

(b) Calculate the radiance of the sun and the radiance of the moon as seen from the earth. Assume the following constants:

$$\text{SolarConstant} = 1367 \text{ Watts m}^{-2}$$

$$\text{SunDiameter} = 1.39 \text{ X } 10^6 \text{ km}$$
$$\text{MoonDiameter} = 3.48 \text{ X } 10^3 \text{ km}$$
$$\text{Sun - EarthDistance} = 1.49 \text{ X } 10^8 \text{ km}$$
$$\text{Sun - MoonDistance} = 1.49 \text{ X } 10^8 \text{ km}$$
$$\text{Earth - MoonDistance} = 3.8 \text{ X } 10^5 \text{ km}$$
$$\text{Reflectivity of the Moon} = 6.7\%$$

Assume that the reflectance from the moon is isotropic (i.e., the moon's surface is said to be a Lambertian reflector).

Problem 2.2

A small perfectly black spherical satellite is in orbit around the earth at a height of 200 km. What solid angle does the earth subtend when viewed from the satellite? **Hint: Consider Fig. 2.6a and assume the Earth's radius to be 6370 km.**

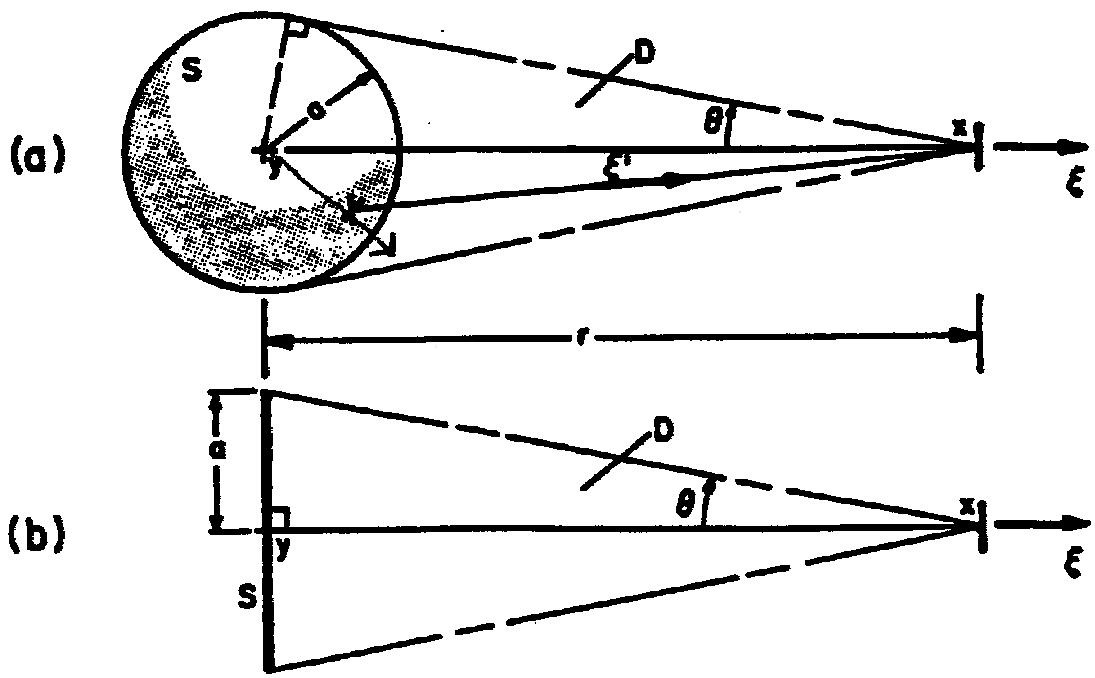

Fig. 2.6 Deriving the irradiance distance-law for spheres and disks.

Problem 2.3

Irradiance Distance Law for Spheres.

Consider a spherical surface S of radius a with uniform radiance distribution of magnitude I at each point. Suppose that S is viewed at a point x a distance r from the center y of S. The lines of sight lie in a vacuum and the background radiance of S is zero. See Fig. 2.6 (a). Derive the irradiance $F(x, \xi)$ at point x in terms of the given variables. Here ξ is the normal at x in the direction from y to x.

Problem 2.4

Irradiance Distance Law for Circular Disks

Refer to Fig. 2.6 (b). That figure depicts a circular disk of radius a of uniform surface intensity I at each point. The disk is viewed at point x on the perpendicular through the center y of S at a distance r from the center. The set D of the lines of sight from x to S lies in a vacuum and the background radiance of S is zero. What is the irradiance $F(x, \xi)$ at point x in terms of the given variables? Here ξ is the same as that in Eqn. (1.3). Compare your answer to that of problem 2.3 by determining the value of r (in units of radius α) such that the difference between the two irradiances is less than 1%.

Problem 2.5

If an incident azimuthally symmetric radiation field is described by $I(\theta) = I_o \tan\theta$, where θ is the zenith angle, briefly describe the visual appearance of such a field and derive an expression between F and I_o.

Problem 2.6

Solve the following:

(a) Using the cosine law and the definition of solid angle in the class notes, establish that the relationship between the hemispheric flux F on a horizontal surface and the intensity I flowing to that surface is

$$F = \int_0^{2\pi} \int_0^{\pi/2} I(\theta,\phi)\cos\theta \sin\theta d\theta d\phi$$

where θ is the zenith angle and ϕ is the azimuthal angle.

(b) Calculate this flux when the intensity field is uniform (isotropic) and flows through a set of directions defined by the angle θ centered on the normal to the horizontal plane. Derive this flux as a function of θ.

(c) Using your results of (b) above, show that the hemisphere flux is πI_o for an isotropic intensity field of magnitude I_o.