Scattered radiation = Scattering event * Incident radiation

$$\begin{bmatrix} E_{\parallel s} \\ E_{\perp s} \end{bmatrix} \propto \underbrace{\begin{bmatrix} S_2 & S_3 \\ S_4 & S_1 \end{bmatrix}}_{\text{“Amplitude scattering matrix”}} \begin{bmatrix} E_{\parallel i} \\ E_{\perp i} \end{bmatrix}$$

More practical approach:
Stokes vector and the phase matrix

$$I \propto \langle E_{\parallel} E_{\parallel}^{*} + E_{\perp} E_{\perp}^{*} \rangle$$

$$Q \propto \langle E_{\parallel} E_{\parallel}^{*} - E_{\perp} E_{\perp}^{*} \rangle$$

$$U \propto \langle E_{\parallel} E_{\perp}^{*} + E_{\perp} E_{\parallel}^{*} \rangle$$

$$V \propto i \langle E_{\parallel} E_{\perp}^{*} - E_{\perp} E_{\parallel}^{*} \rangle$$

$$\begin{bmatrix} I_s \\ Q_s \\ U_s \\ V_s \end{bmatrix} = \frac{C_{sca}}{4\pi r^2} \underbrace{\begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix}}_{} \begin{bmatrix} I_i \\ Q_i \\ U_i \\ V_i \end{bmatrix}$$

Phase matrix depends on particle size, shape, composition, orientation, wavelength of light, illumination geometry

Note:
$\sigma_{sca} = C_{sca}$

Phase matrix is simplified in special cases:

Rayleigh scattering

$$\begin{bmatrix} P_{11} & P_{12} & 0 & 0 \\ P_{12} & P_{11} & 0 & 0 \\ 0 & 0 & P_{33} & 0 \\ 0 & 0 & 0 & P_{33} \end{bmatrix}$$

Mie scattering (= spheres)

$$\begin{bmatrix} P_{11} & P_{12} & 0 & 0 \\ P_{12} & P_{11} & 0 & 0 \\ 0 & 0 & P_{33} & P_{34} \\ 0 & 0 & -P_{34} & P_{33} \end{bmatrix}$$

$$\begin{bmatrix} P_{11} & P_{12} & 0 & 0 \\ P_{12} & P_{22} & 0 & 0 \\ 0 & 0 & P_{33} & P_{34} \\ 0 & 0 & -P_{34} & P_{44} \end{bmatrix}$$

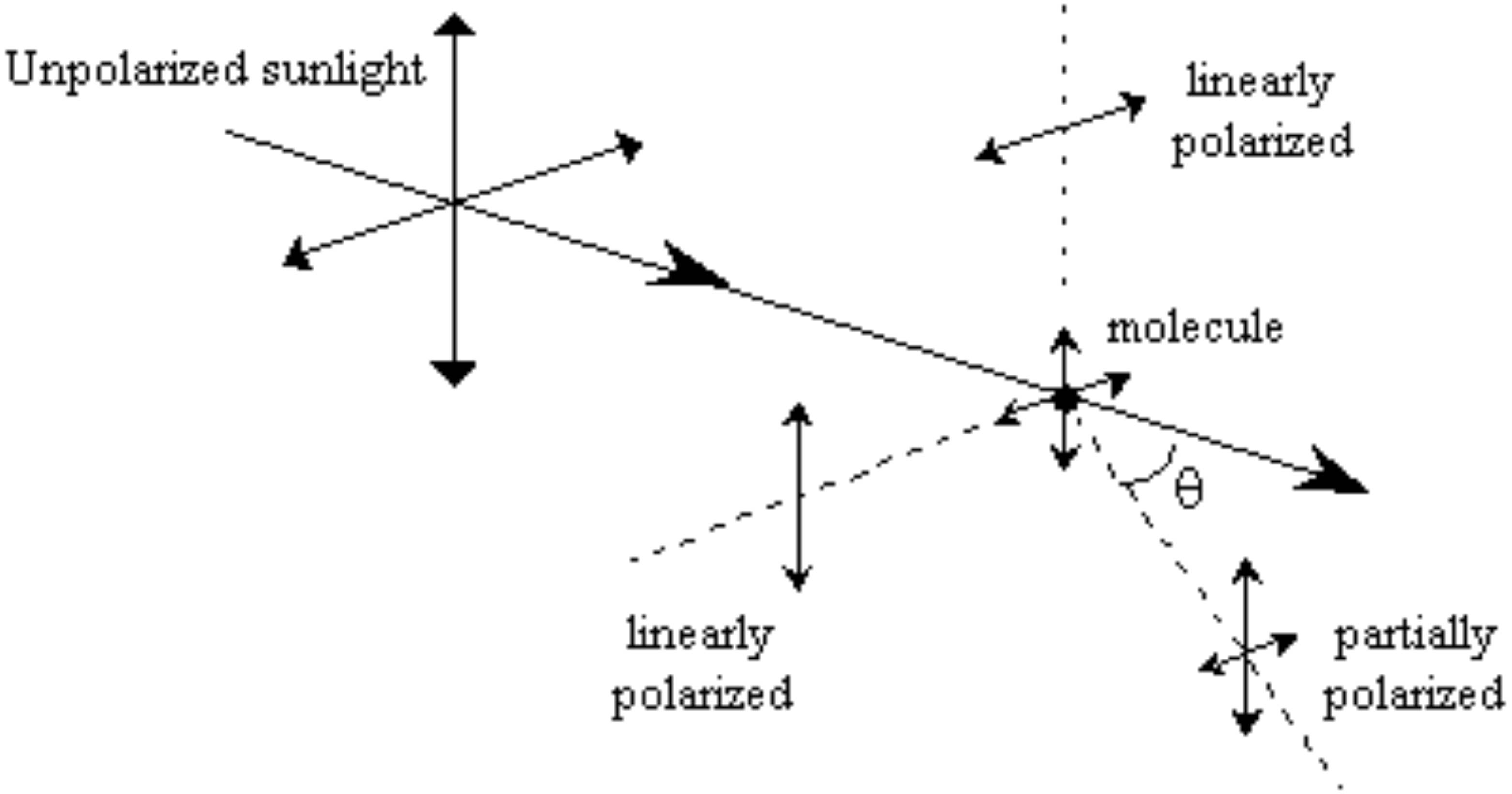
More general case. Randomly oriented nonspherical particles, equal amount of particles and their mirror particles.

If incident light is *unpolarized* (*Q,U,V* = 0):

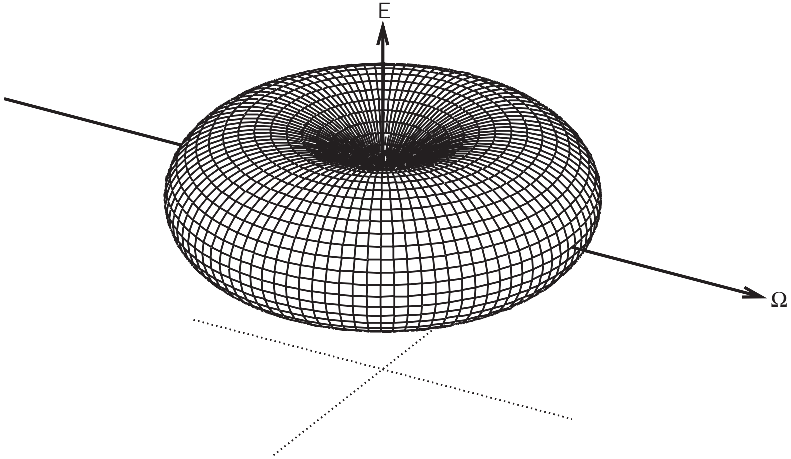
Intensity of scattered light $I_s \propto P_{11}(\Theta)$

Degree of linear polarization $-\dfrac{P_{12}(\Theta)}{P_{11}(\Theta)}$

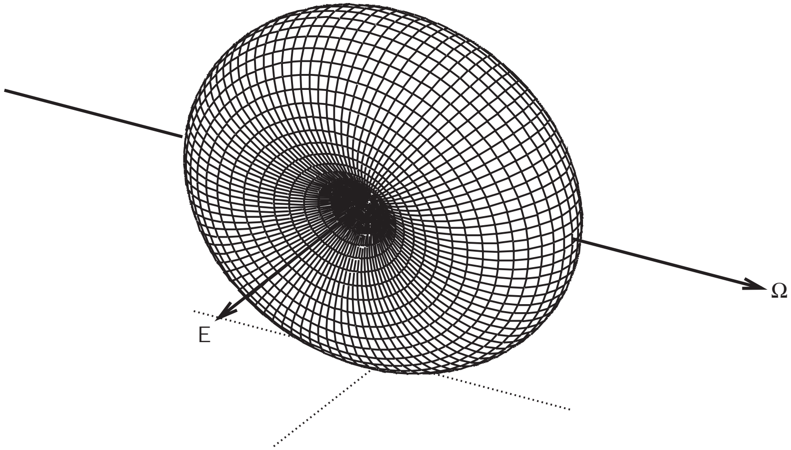
Rayleigh Scattering Geometry



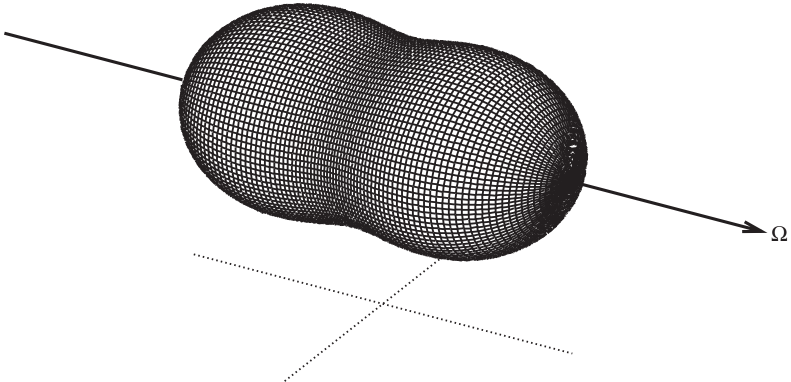
Vertical Incoming Polarization



Horizontal Incoming Polarization



Incident Light Unpolarized



Rayleigh Scattering formulae

$$Q_s = \frac{8}{3} x^4 \left| \frac{m^2 - 1}{m^2 + 2} \right|^2$$

$\varpi \sim x^3$

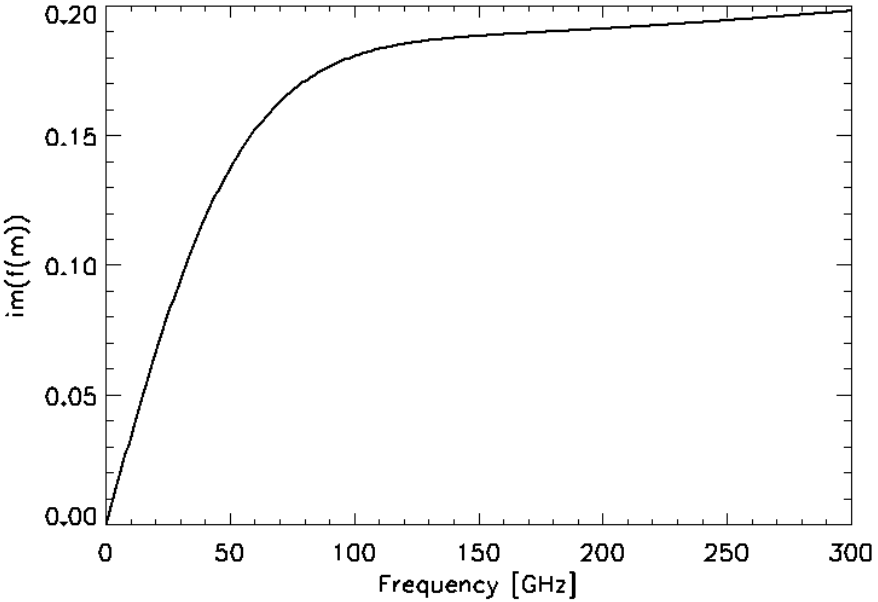
$$Q_a = 4x \operatorname{Im}\left(\frac{m^2 - 1}{m^2 + 2} \right)$$

Phase Matrix

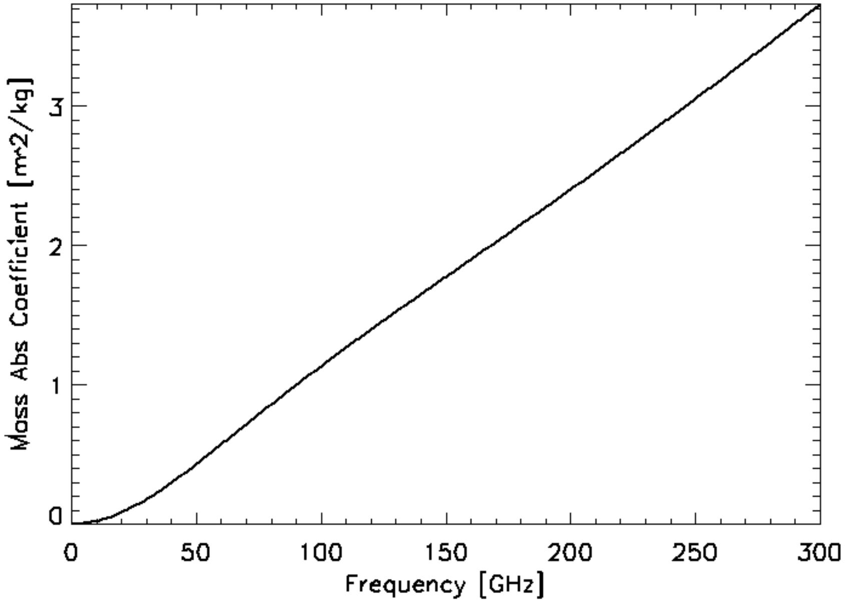
$$\begin{bmatrix} 1 + \cos^2\Theta & \cos^2\Theta - 1 & 0 & 0 \\ \cos^2\Theta - 1 & 1 + \cos^2\Theta & 0 & 0 \\ 0 & 0 & 2\cos\Theta & 0 \\ 0 & 0 & 0 & 2\cos\Theta \end{bmatrix}$$

Cloud Water Absorption in the microwave

$$\mathrm{Im}\left(\frac{m^2-1}{m^2+2}\right)$$



k (mass abs. coeff)



Mie solution; the result

$$Q_\mathrm{e} = \frac{2}{x^2}\sum_{n=1}^{\infty}(2n+1)\Re(a_n+b_n)\,, \tag{12.23}$$

$$S_1 = \sum_n \frac{2n+1}{n(n+1)}(a_n\pi_n + b_n\tau_n)$$

$$Q_\mathrm{s} = \frac{2}{x^2}\sum_{n=1}^{\infty}(2n+1)(|a_n|^2+|b_n|^2)\,, \tag{12.24}$$

$$S_2 = \sum_n \frac{2n+1}{n(n+1)}(a_n\tau_n + b_n\pi_n)$$

Expansion coefficients of the scattered field:

$$a_n = \frac{m\psi_n(mx)\psi_n'(x) - \psi_n(x)\psi_n'(mx)}{m\psi_n(mx)\xi_n'(x) - \xi_n(x)\psi_n'(mx)}$$

$$b_n = \frac{\psi_n(mx)\psi_n'(x) - m\psi_n(x)\psi_n'(mx)}{\psi_n(mx)\xi_n'(x) - m\xi_n(x)\psi_n'(mx)}$$

The series expansions converge at large *n* and are, in practice, truncated at some *n* ≈ *x* … 2*x*.

Here *ψ*, *ξ* are Riccati-Bessel functions and *π, τ* are functions of Legendre polynomials, *x* is the size parameter and *m* the complex refractive index.

Mie solution; using the result

Typical Inputs:

x (size parameter)
m (complex refractive index)

Typical Outputs:

$S_1(\Theta)$, $S_2(\Theta)$: Amplitude scattering matrix (complex)
Qe, Qs : Ext, Scat efficiencies

$$\sigma_e = Q_e \pi r^2$$

$$\sigma_s = Q_s \pi r^2$$

$$\beta = k\rho = \sigma n$$

$$\rho = \frac{4}{3}\pi r^3 \rho_m n$$

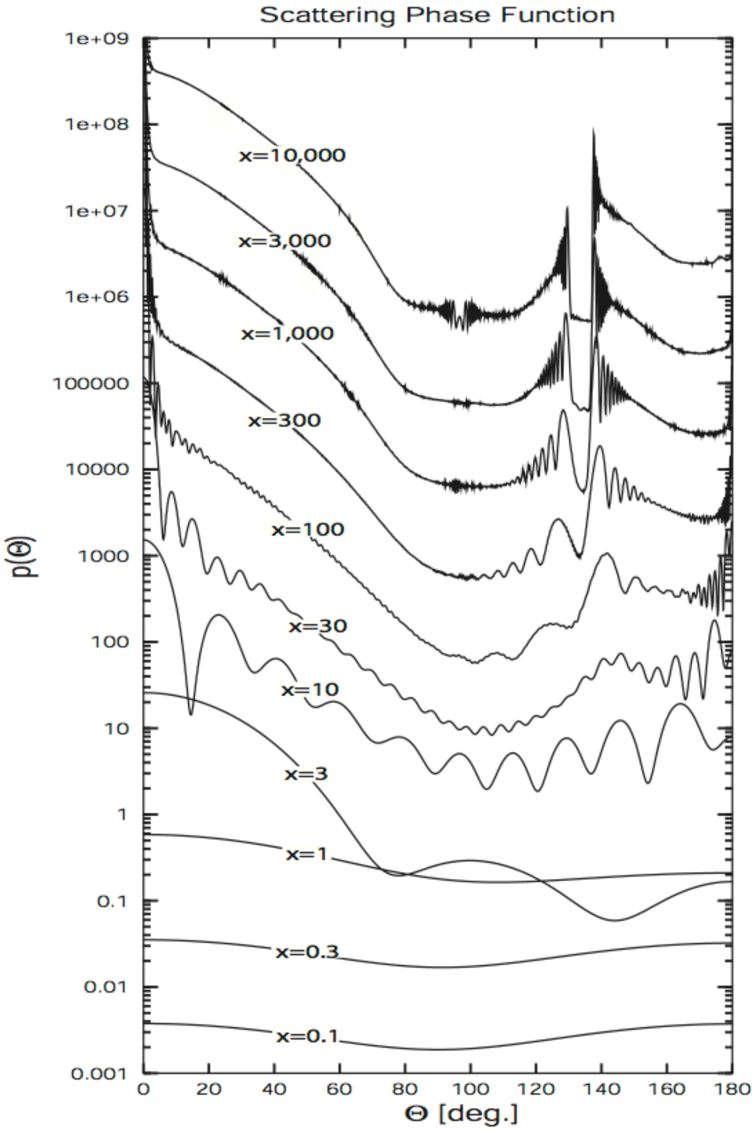
$$P_{11} = \frac{2}{Q_s x^2}\left(\left|S_1\right|^2 + \left|S_2\right|^2\right)$$

$$P_{12} = \frac{2}{Q_s x^2}\left(\left|S_2\right|^2 - \left|S_1\right|^2\right)$$

$$P_{33} = \frac{2}{Q_s x^2}\left(S_2^* S_1 + S_1^* S_2\right)$$

$$P_{34} = \frac{2}{Q_s x^2}\,\mathrm{Im}\left(S_2^* S_1 - S_1^* S_2\right)$$

Phase Function of water spheres (Mie theory)



High Asymmetry Parameter

Low Asymmetry Parameter

Mie Theory

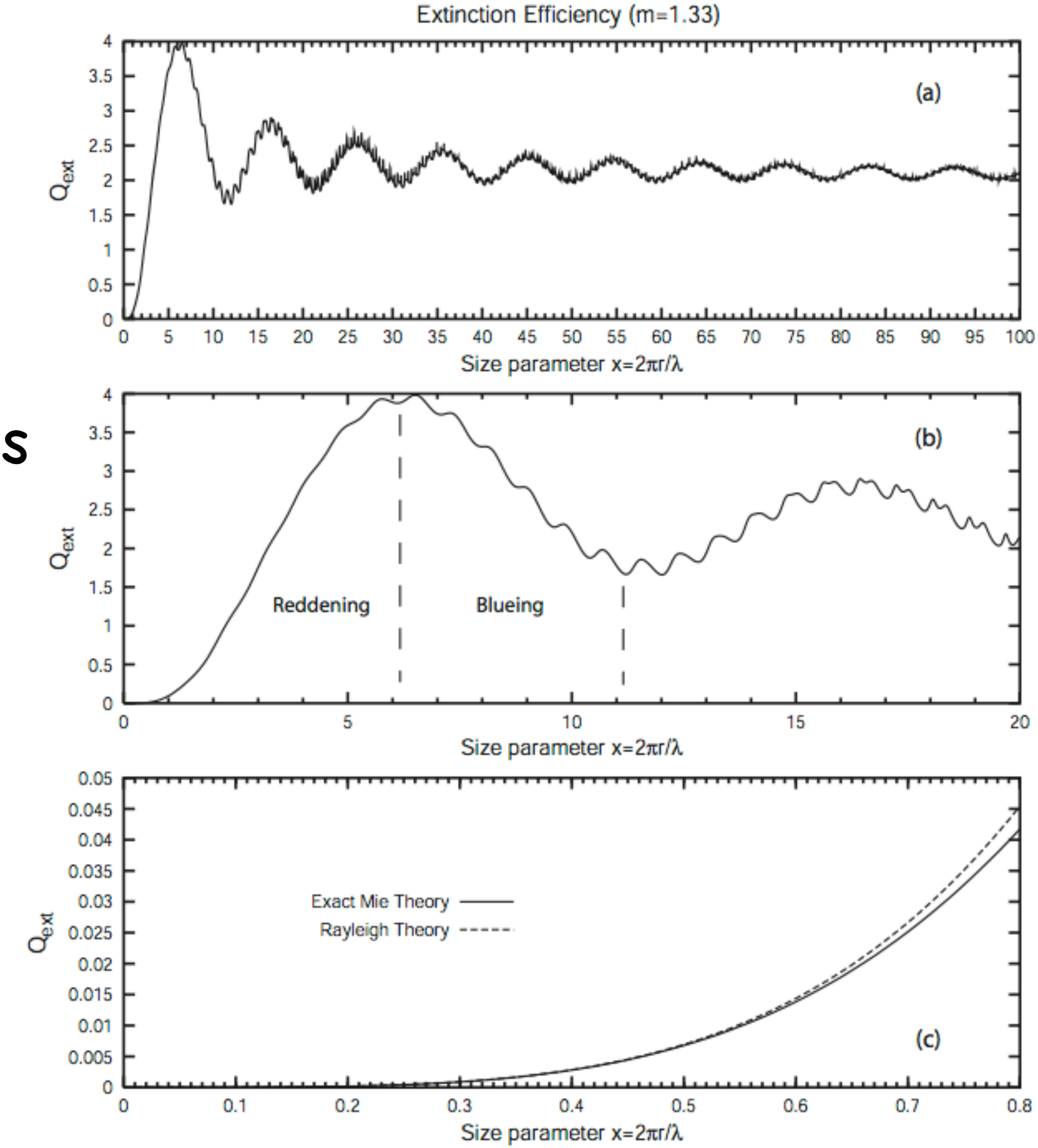
- Exact Qs, Qa for spheres of some x, m.

$$Q_{\mathrm{e}} = \frac{2}{x^2} \sum_{n=1}^{\infty} (2n+1) \Re(a_n + b_n) \,, \qquad (12.23)$$

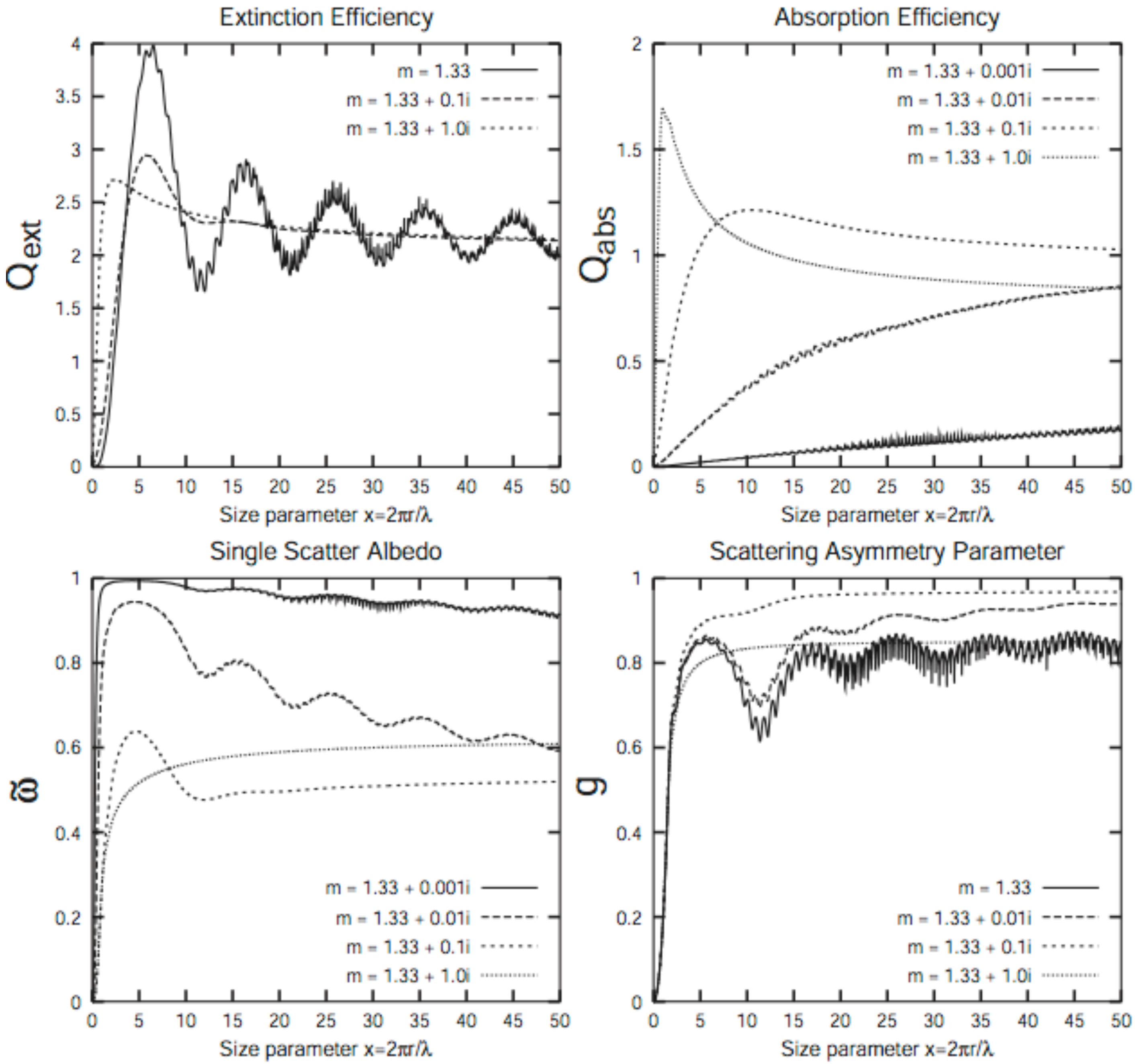
$$Q_{\mathrm{s}} = \frac{2}{x^2} \sum_{n=1}^{\infty} (2n+1) (|a_n|^2 + |b_n|^2) \,, \qquad (12.24)$$

- a, b coefficients are called “Mie Scattering coefficients”, functions of x & m. Easy to program up.
- “bhmie” is a standard code to calculate Q-values in Mie theory.
- Need to keep approximately $x + 4x^{1/3} + 2$ terms for convergence

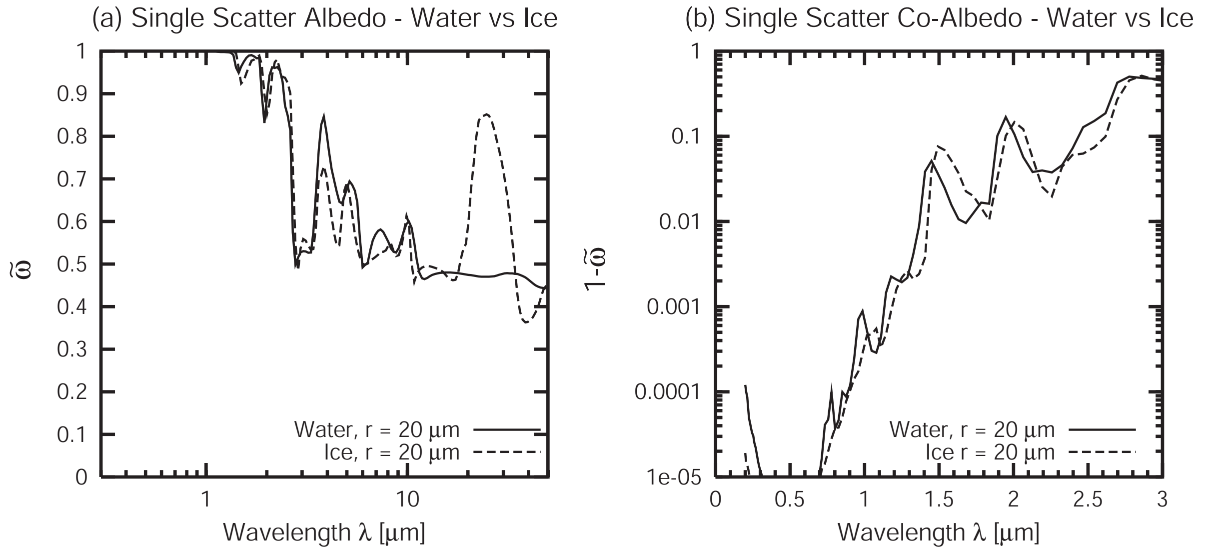
Q_e for NON-ABSORBING SPHERES



Mie Theory Results for ABSORBING SPHERES

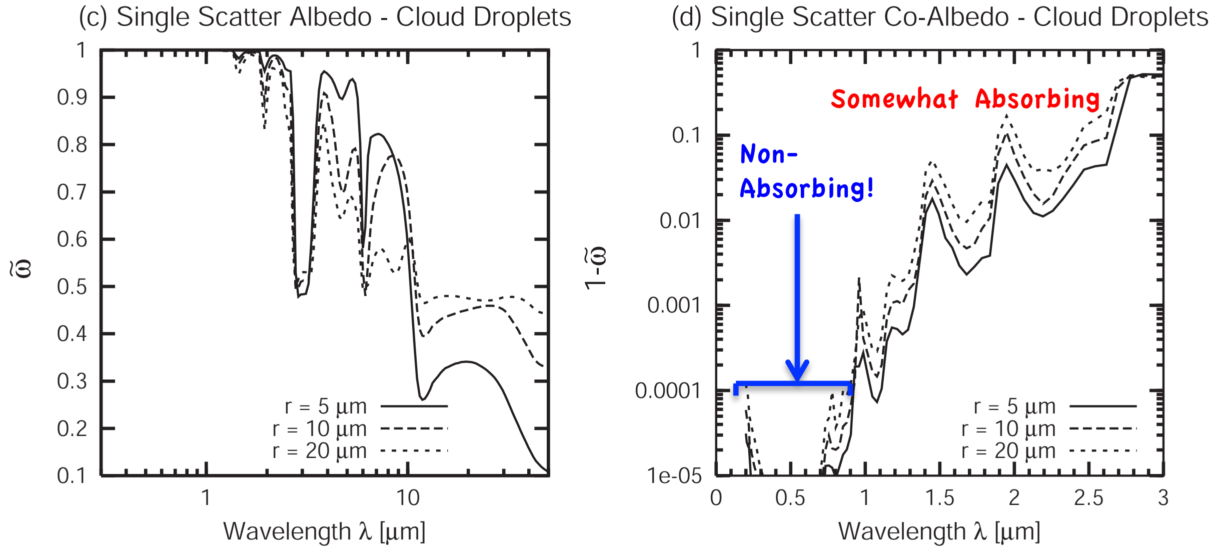


Variations of SSA with wavelength



(a) Single Scatter Albedo - Water vs Ice

(b) Single Scatter Co-Albedo - Water vs Ice



(c) Single Scatter Albedo - Cloud Droplets

(d) Single Scatter Co-Albedo - Cloud Droplets

Somewhat Absorbing

Non-Absorbing!