

# How to construct an Adding/Doubling Radiative Transfer Model for thermal and solar sources

IMMEDIATE

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## Overview

This project is organized into two primary pieces.

**Section 1: Construct a scalar (intensity-only) adding-doubling model with only thermal sources** This model will:

1. Take as input optical depth, single scattering albedo, and phase function expansion coefficients for  $N$  layers in the atmosphere.
2. Take as input the surface reflectance, which is assumed to be Lambertian.
3. Take as input temperature at the  $N + 1$  layer boundaries. You will assume temperature varies linear with  $\tau$  across a layer.
4. Calculate the downwelling radiance (or brightness temperature  $T_B$ ) at the surface, and the upwelling radiance (or  $T_B$ ) at TOA, at a user-specified view zenith angle  $\theta_u$
5. Because of the azimuthal symmetry, we need not specify the viewing azimuth angle. This will change in Part 2, when we add a solar source.

**Section 2: Modify the RT model in Part I to include a solar source.** To accomplish this, two primary modifications are necessary:

1. Add an outer loop to the RT program to loop over azimuthal moments  $m$ .
2. Add a solar source to each initial thin layer before doubling.
3. Include additional doubling equations for the solar source.
4. Include additional surface solar source term

# 1 Adding-Doubling Model with Thermal Sources

We will construct the model in pieces, then put them all together. When coding it, it is recommended to make each piece be a module. For instances, modules will be used to:

1. Calculate the quadrature points and weights
2. Construct the phase function matrices
3. Doubling to determine the  $\mathbf{R}$  &  $\mathbf{T}$  matrices and source vectors for each layer.  
- This requires thin layer initialization and doubling equations.
4. Adding to determine the  $\mathbf{R}$  &  $\mathbf{T}$  matrices and source vector for the entire atmosphere
5. Construct the surface reflectancematrix  $\mathbf{R}_g$  and surface upwelling source vector  $\mathbf{s}_g^+$
6. Combine the atmosphere with the surface, plus the incoming CMB radiance.
7. Inverse Planck function Module to determine  $T_B = B_\lambda^{-1}(I_\lambda)$

You must write a piece of code that will accept inputs and calculate outputs. You may write this in any language you like, but something like C or Fortran will generally be much faster than an interpreted language like MatLab or IDL.

## 1.1 Quadrature Points and Weights

The basic way quadrature schemes work is to calculate radiative transfer for specific upwelling and downwelling zenith directions called *streams*. Usually, one would have to interpolate to any desired zenith direction from the chosen streams. However, it is preferred to add a “stream with zero weight” to the list of streams, for the zenith angle  $\mu_u = \cos \theta_u$  at the desired zenith angle  $\theta_u$ . It will not affect the calculations for the streams with weight, but it will provide a much more accurate radiance answer at the desired zenith angle, without the need for any additional interpolation.

Begin by determining the quadrature points  $\boldsymbol{\mu}$  and weights  $\mathbf{w}$ . Note that throughout this document, we use the convention that a capital boldface letter (such as  $\mathbf{R}$ ) denotes a matrix, whereas a lower-case boldface letter (such as  $\mathbf{w}$ ) denotes a vector. Use some version of “gauleg.f” or “gauleg.pro” provided. I suggest you use a double-gauss quadrature scheme. Call it on the range  $0 \dots 1$ . This will produce  $\mu_1, \mu_2, \dots \mu_{N_s}$ , where  $N_s$  is the number of streams in a hemisphere (sometimes called the number of “half-streams”). It will also give you the corresponding weights  $w_1, w_2, \dots w_{N_s}$ .

Next, construct the quadrature points vector  $\boldsymbol{\mu}$  and weights vector  $\mathbf{w}$ , including at the

bottom the user-desired stream with zero weight:

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{N_s} \\ \mu_u \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N_s} \\ 0 \end{pmatrix} \quad (1)$$

Notice each of these has  $N_s + 1$  elements. The weights vector  $\mathbf{w}$  will have the property that its elements sum to unity. In the next several sections, we will calculate many square matrices that will be based on these, and will all have  $(N_s + 1, N_s + 1)$  elements. Note that in the usual nomenclature,  $N_s = 1$  corresponds to a 2-stream model,  $N_s = 2$  a 4-stream model, etc, because usually one talks about the total number of streams (up plus down).

## 1.2 Calculation of Phase Function Matrices

In this section we'll describe how to specify the phase function matrix for each layer. This is simply a discretized form of the phase function, but after we have rewritten the radiative transfer equation for each azimuthal moment  $m$ . Here we will set  $m = 0$ , but later we'll need to write it for any  $m$ , so the in the formula below we'll keep do it for arbitrary  $m \geq 0$ . We'll construct two matrices; one for radiation that gets scattered *forwards*,  $\mathbf{P}_m^+$ , and one for radiation scattered *backwards*,  $\mathbf{P}_m^-$ . Note that from here on out, we'll generally think of all zenith angles as  $\mu \geq 0$ , and just remember to put in the  $-$  sign for downwelling radiation only in those places where it is necessary. This will happen below, and I'll point out exactly where.

$\mathbf{P}^+$  ( $\mathbf{P}^-$ ) represents the intensity phase function matrix for scattering into the forward (background) direction.  $\mathbf{P}_m^+(i, j)$ , which represents scattering from incident stream  $j$  into emergent stream  $i$  in the forward direction for azimuthal mode  $m$ , is calculated according to

$$P_m^+(i, j) = \sum_{\ell=m}^L a_\ell^m \chi_\ell P_\ell^m(\mu_i) P_\ell^m(\mu_j) \quad (2)$$

where  $\chi_\ell$  is the  $\ell^{\text{th}}$  phase function expansion coefficient,  $P_\ell^m(\mu)$  represents the associated legendre polynomial,  $a_\ell^m$  is a pre-factor defined as

$$a_\ell^m = (2 - \delta_{0,m}) \frac{(\ell - m)!}{(\ell + m)!} \quad (3)$$

(Note that  $a_\ell^0 = 1$ .) Similarly,  $\mathbf{P}_m^-$  is calculated as  $\mathbf{P}_m^+$  upon substituting  $\mu_i \rightarrow -\mu_i$ . Noting that  $P_\ell^m(-\mu) = (-1)^{\ell+m} P_\ell^m(\mu)$ , we find that

$$P_m^-(i, j) = \sum_{\ell=m}^L (-1)^{\ell+m} a_\ell^m \chi_\ell P_\ell^m(\mu_i) P_\ell^m(\mu_j) \quad (4)$$

The choice of  $L$  is not straightforward. One has two choices: to set it to a large enough value such that the expansion coefficients faithfully capture the full phase function behavior, or set it to some lower value. The reason for the latter are that a small number of

hemispheric streams  $N_s$  cannot resolve the phase function anyway. A particularly useful choice is  $L = 2N_s - 1$ ; this choice does not require a renormalization step (see below), and often provides better results, even though some elements of  $\mathbf{P}_m^+$  and  $\mathbf{P}_m^-$  may be negative.

Note that for  $m = 0$ , all we have done is to calculate the *azimuthally-averaged phase function value*,  $\tilde{P}(\mu_i, \mu_j)$ , for scattering from incident zenith angle  $\mu_j$  to emergent zenith angle  $\mu_i$ , assuming  $L$  is suitably large to capture the full phase function.

Both of these matrices  $\mathbf{P}_m^+$  and  $\mathbf{P}_m^-$  must be calculated for each layer in the model atmosphere, where you have pre-calculated the phase function expansion coefficients from the total phase function in each layer. Note that these matrices will be symmetric; that is,  $P_m^+(i, j) = P_m^+(j, i)$  and similarly for  $\mathbf{P}_m^-$ .

Note that it is a good idea to ensure normalization for  $m = 0$ . This will not always be the case, even with a properly normalized input phase function. We require that the quadrature-weighted sum over outgoing angles  $i$  for a given incoming angle  $j$  add to 2:

$$\sum_{i=1}^{Ns+1} (P_0^+(i, j) + P_0^-(i, j))w_i = 2. \quad (5)$$

To ensure this normalization, one follows the following steps:

$$f_{norm}(j) = \frac{1}{2} \sum_{i=1}^{Ns+1} (P_0^+(i, j) + P_0^-(i, j))w_i \quad (6)$$

$$P_0^\pm(i, j) = P_0^\pm(i, j)/f_{norm}(j) \quad (7)$$

for each incoming direction  $j$ . Note that this is simply one form of a possible renormalization - there are many other possibilities depending on how the corrections are distributed within the elements of each column  $j$  of the phase matrix. Wiscombe (1976) discusses a number of approaches. Regarding this “renormalization step”, I note two things. First, if it requires this step, then the resulting matrix will *no longer be symmetric in general*, though some methods will retain the symmetric shape, e.g. that of Grant (described in Wiscombe (1976)). Second, the conditions under which it will be properly normalized (and not require this extra step) are:

- $L$  is set to something large, but the phase function expansion coefficients  $\chi_\ell$  have basically died away by  $\ell = 2 * N_s$ . **OR**
- $L$  is set to  $L = 2N_s - 1$  in Equations (2) and (4), and Gaussian quadrature on (0,1) is employed.

### 1.3 Doubling: Calculation of layer Reflection & Transmission matrices and Source Functions

In this section we will describe how to calculate the reflectance matrix  $\mathbf{R}$  for back-scattering from an incident stream  $j$  to an emergent stream  $i$ , as well as the transmittance

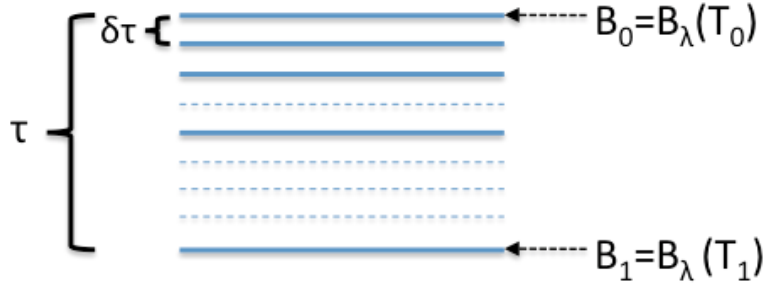


Figure 1: Example of doubling: a layer with optical depth  $\tau$  is divided into 8 identical thin layers; in this case  $d = 3$  doublings will generate the full layer from the initial layers.

matrix  $\mathbf{T}$  which describes *both* direct transmission as well as diffuse transmission. The latter means forward-scattering from incident stream  $j$  to an emergent stream  $i$ . In the same calculation, we can compute the upwelling and downwelling *source functions* of the layer,  $\mathbf{s}^+$  and  $\mathbf{s}^-$ .

Due to the variation of temperature (and potentially the solar source) across the layer, the upwelling and downwelling sources will generally be different. However, because the phase function and single-scattering albedo  $\tilde{\omega}$  are assumed to be constant across the layer, the matrices  $\mathbf{R}$  and  $\mathbf{T}$  will be the same for incident radiation from above or below the layer. Note that this generally will *not* be true once we add together layers with different optical properties.

The method we will use to accomplish this is referred to as *doubling*. The modern doubling method was laid out by Warren Wiscombe in the mid 1970s (Wiscombe, 1976, 1975, 1977a). It is accurate for fairly thin layers, but can be slow for layers with large scattering optical thicknesses. In that case, alternate methods are preferred (see e.g. McGarragh and Gabriel, 2010).

In doubling, we will begin with a small optical depth  $\delta\tau$  which is obtained by  $d$  halvings of the layer with initial optical depth  $\tau$ :  $\delta\tau = \tau/2^d$ . Figure 1 shows an example of this for  $d = 3$ . I recommend choosing  $d$  for a given layer  $l$  such that  $\delta\tau < 0.01$ . It is good to make this maximum initial layer thickness a free parameter to your RT model, so you can easily vary it.

### 1.3.1 Thin-Layer initialization

Consider a very thin layer with optical thickness  $\delta\tau$  single scattering albedo  $\tilde{\omega}$  and phase function matrices given by  $\mathbf{P}^+$  and  $\mathbf{P}^-$ . The reflection and transmission matrices for this thin layer will be denoted  $\mathbf{R}_n$  and  $\mathbf{T}_n$ , respectively, with upwelling and downwelling source vectors  $\mathbf{s}_0^+$  and  $\mathbf{s}_0^-$ , respectively. We will also assume that the planck function  $B(T(\tau))$  varies linearly in  $\tau$ . This turns out to be a reasonably justified assumption in many cases (and is much better than assuming that the temperature is constant across the entire layer). Assume the layer has temperature  $T_0$  at the lower boundary and  $T_1$  at

the upper boundary. Then it will have corresponding Planck functions  $B_0 = B_\lambda(T_0)$  and  $B_1 = B_\lambda(T_1)$ . Next define  $B_d$  as the slope of  $B$  across the layer:

$$B_d \equiv \frac{B_0 - B_1}{\tau} \quad . \quad (8)$$

Wiscombe (1976) and Heidinger et al. (2006) explore different initialization schemes. Here, we will use the simplest of all initialization schemes, called the *infinitesimal generator initialization* scheme (IGI). Let the letter  $k$  correspond to the number of times our layer has been doubled. When  $k = 0$ , this is the initial layer of optical thickness  $\delta\tau$ . The reflection and transmission matrices for this thin layer are given by:

$$R_0(i, j) = (1 + \delta_{0,m}) \frac{\tilde{\omega} \delta\tau}{4 \mu_i} P_m^-(i, j) w_j \quad (9a)$$

$$T_0(i, j) = (1 + \delta_{0,m}) \frac{\tilde{\omega} \delta\tau}{4 \mu_i} P_m^+(i, j) w_j + \delta_{i,j} (1 - \delta\tau/\mu_i) \quad (9b)$$

where the second term on the right-hand side of (9b) accounts for the direct transmission term. Again, for azimuthal symmetry (such as a pure thermal source or pure flux calculations), the azimuthal moment  $m$  will equal 0. In section 2, we'll use these exact same equations but with  $m \geq 0$ .

Accounting for a varying linearly-varying thermal source term across the full layer is slightly more complicated. The formulation for this is derived in Wiscombe (1975). For each layer, there will be a term related to the mean layer source  $\mathbf{y}\bar{B}$ , and a term related to the linear variation of the source across the layer  $\mathbf{z}B_d$ . Therefore,  $\mathbf{y}$  and  $\mathbf{z}$  can be thought of as layer emissivity and emissivity slope in a given (outgoing) direction  $\mu_i$ . For the initial (very thin) layer at doubling  $k = 0$ , they are given by

$$\mathbf{y}_0 = (1 - \tilde{\omega}) \frac{\delta\tau}{\boldsymbol{\mu}} \delta_{0,m} \quad (10a)$$

$$\mathbf{z}_0 = 0 \quad . \quad (10b)$$

The Kronecker-delta  $\delta_{0,m}$  is included to make explicit that the thermal source *only* contributes to the azimuthally-average source. For  $m > 0$ , as will occur in section 2 with the additional of a solar source, the thermal contribution is 0.

### ***A note about the Planck function:***

For microwave radiation, one can often get away with working in brightness temperature space the entire time, rather than working in intensity and calculating brightness temperature at the very end. However, the latter is generally safer unless you know that the Rayleigh-Jeans approximation is valid.

As a reminder, the Planck function is given by:

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5 (e^{\frac{hc}{\lambda k_B T}} - 1)} \quad [W m^{-2} \mu m^{-1} sr^{-1}] \quad (11)$$

where  $\lambda$  is the wavelength given in  $\mu m$ ,  $T$  is given in K, and the following two constants should be used:

$$hc^2 = 5.9552136 \cdot 10^7 [Wm^{-2}\mu m^4] \quad (12a)$$

$$hc/k_B = 14387.752 [K\mu m] \quad (12b)$$

The inverse Planck Function (determining brightness temperature  $T_B$  from a radiance  $I_\lambda$ ) is similarly given by:

$$T_B = B_\lambda^{-1}(I_\lambda) = \frac{hc/k_B}{\lambda \ln\left(\frac{2hc^2}{\lambda^5 I_\lambda} + 1\right)} \quad (13)$$

### 1.3.2 The Doubling Equations with a thermal source

Using the principle of interaction, it is straightforward to derive the equations to double two adjacent layers with a linearly-varying thermal source (Wiscombe, 1975). To go from doubling  $k$  to doubling  $k + 1$ , they are:

$$\mathbf{R}_{k+1} = \mathbf{T}_k \mathbf{\Gamma}_k \mathbf{R}_k \mathbf{T}_k + \mathbf{R}_k \quad (14a)$$

$$\mathbf{T}_{k+1} = \mathbf{T}_k \mathbf{\Gamma}_k \mathbf{T}_k \quad (14b)$$

$$\mathbf{z}_{k+1} = (\mathbf{T}_k \mathbf{\Gamma}_k - \mathbf{T}_k \mathbf{\Gamma}_k \mathbf{R}_k)(\mathbf{z}_k - g_k \mathbf{y}_k) + g_k \mathbf{y}_k + \mathbf{z}_k \quad (14c)$$

$$\mathbf{y}_{k+1} = (\mathbf{T}_k \mathbf{\Gamma}_k + \mathbf{T}_k \mathbf{\Gamma}_k \mathbf{R}_k + \mathbf{I}) \mathbf{y}_k \quad (14d)$$

where  $\mathbf{I}$  is the identity matrix, and  $g_k \equiv 2^{k-1} \delta\tau$  ( $g_0 = 0.5\delta\tau$ ,  $g_1 = \delta\tau$ ,  $g_2 = 2\delta\tau$ , etc), and

$$\mathbf{\Gamma}_k \equiv (\mathbf{I} - \mathbf{R}_k^2)^{-1} \quad (15)$$

Equations (14a)-(14d) are to be repeated until the  $k = d$  terms are obtained (that is, a loop from  $k = 0$  to  $k = d$ ). To obtain the upwelling and downwelling thermal sources of the layer  $\mathbf{s}^+ = \mathbf{s}_d^+$  and  $\mathbf{s}^- = \mathbf{s}_d^-$ , we must apply the calculated mean emissivity  $\mathbf{y}_d$  and emissivity slope  $\mathbf{z}_d$  of the layer as follows:

$$\mathbf{s}_d^+ = \mathbf{y}_d \bar{B} + B_d \mathbf{z}_d \quad (16a)$$

$$\mathbf{s}_d^- = \mathbf{y}_d \bar{B} - B_d \mathbf{z}_d \quad (16b)$$

where  $\bar{B} = (B_0 + B_1)/2$  is the mean planck function for the layer. Note that if the source does not vary across the layer, then  $\mathbf{s}^+ = \mathbf{s}^-$  as we know must happen.

## 1.4 Adding: Put together the whole atmosphere

At this point, we (should) know the Reflection matrix  $\mathbf{R}_l$ , the transmission matrix  $\mathbf{T}_l$ , and the upwelling and downwelling thermal sources  $\mathbf{s}_l^+$  and  $\mathbf{s}_l^-$  for each layer  $l$  in the atmosphere<sup>1</sup>. The adding formula are straightforward to derive for two layers with different properties. They are very similar to the doubling formulae, but we have the added

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<sup>1</sup>Note that  $l$  here is used to denote layer index, should not be confused with the *cursive*  $\ell$  we used early to sum over Phase function expansion coefficients.

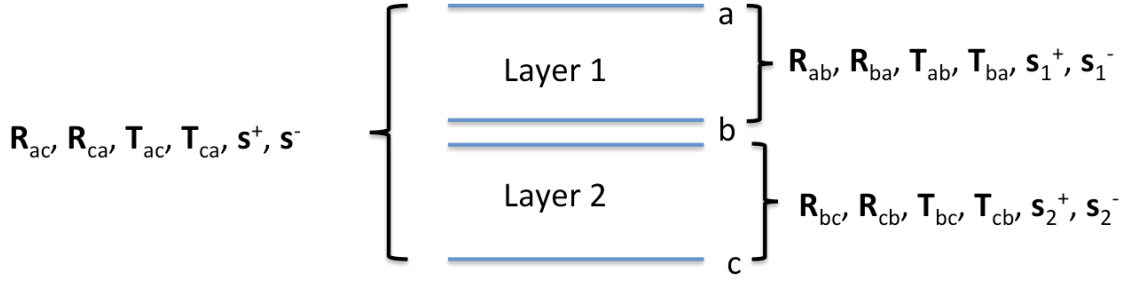


Figure 2: Representation of the adding process of two adjacent, inhomogenous layers.

complication that, for a layer in general, the matrices  $\mathbf{R}$  and  $\mathbf{T}$  may be different for light incident from above vs. light incident from below. For our doubled layers, this is not the case -  $\mathbf{R}$  and  $\mathbf{T}$  are the same whether light is incident from below or above. This is because the optical properties were assumed to be unchanging throughout the layer. This is represented in Figure 2, which depicts two adjacent inhomogenous layers. Layer 1 is the upper layer, and is bounded above by  $a$  and below by  $b$ . Layer 2 is bounded above by  $b$  and below by  $c$ . The six optical property quantities of each are given in the figure (reflection matrices for light incident above and below, transmission matrices for light incident above and below, and upwelling source at the top and downwelling source at the bottom of the layer). For example,  $\mathbf{R}_{ab}$  means the reflection matrix for layer  $ab$  for light incident above (hence going in the direction from  $a$  to  $b$ ). And so on.

In general we need 6 adding formulae, one for each of these six properties. These are given below. Please note that in coding these, many of the calculations are repeated, and the matrix inverse function is the more computationally expensive, so it is good to save repeated intermediate calculations.

$$\mathbf{R}_{ac} = \mathbf{R}_{ab} + \mathbf{T}_{ba}(\mathbf{I} - \mathbf{R}_{bc}\mathbf{R}_{ba})^{-1}\mathbf{R}_{bc}\mathbf{T}_{ab} \quad (17a)$$

$$\mathbf{R}_{ca} = \mathbf{R}_{cb} + \mathbf{T}_{bc}(\mathbf{I} - \mathbf{R}_{ba}\mathbf{R}_{bc})^{-1}\mathbf{R}_{ba}\mathbf{T}_{cb} \quad (17b)$$

$$\mathbf{T}_{ac} = \mathbf{T}_{bc}(\mathbf{I} - \mathbf{R}_{ba}\mathbf{R}_{bc})^{-1}\mathbf{T}_{ab} \quad (18a)$$

$$\mathbf{T}_{ca} = \mathbf{T}_{ba}(\mathbf{I} - \mathbf{R}_{bc}\mathbf{R}_{ba})^{-1}\mathbf{T}_{cb} \quad (18b)$$

$$\mathbf{s}^+ = \mathbf{s}_1^+ + \mathbf{T}_{ba}(\mathbf{I} - \mathbf{R}_{bc}\mathbf{R}_{ba})^{-1}(\mathbf{s}_2^+ + \mathbf{R}_{bc}\mathbf{s}_1^-) \quad (19a)$$

$$\mathbf{s}^- = \mathbf{s}_2^- + \mathbf{T}_{bc}(\mathbf{I} - \mathbf{R}_{ba}\mathbf{R}_{bc})^{-1}(\mathbf{s}_1^- + \mathbf{R}_{ba}\mathbf{s}_2^+) \quad (19b)$$

The easiest is probably to start from the top of the atmosphere, and keep adding layers below until you've gone all the way to the surface. In this way you'll have added together all the layers of the atmosphere into a single layer with known transmission & reflection matrices and upwelling & downwelling source vectors.



## 1.5 Construction of the surface reflectance matrix and upwelling source

Similarly for the atmosphere, in general we can specify the optical properties of the surface in this quadrature land with a reflectance matrix  $R_g$  and an upwelling source vector  $\mathbf{s}_g^+$ . We assume that photons transmitted through the surface interface never return. For construction of these objects, the key quantity is the surface emissivity. In our simple model we will assume either a specular or a Lambertian surface. For land, let us assume Lambertian, and for ocean let us assume specular. In the microwave, the ocean is often treated as specular, though it is typically treated as Lambertian in the thermal infrared. Land surfaces in the thermal IR and microwave are virtually always treated as Lambertian, and typically in the shortwave (UV-Vis-NIR) as well.

### 1.5.1 Lambertian surface optical properties

For a Lambertian surface with surface emissivity  $\epsilon$ , the reflectance (or albedo) is given by  $\alpha_{LW} = 1 - \epsilon$ . I've include the "LW" subscript to be clear that this is a longwave albedo, which is specifically differentiated from a shortwave albedo. The surface reflectance matrix  $\mathbf{R}_g$  and surface source vector  $\mathbf{s}_g^+$  are given by:

$$\mathbf{R}_g = 2\alpha_{LW} \begin{pmatrix} \mu_1 w_1 & \mu_2 w_2 & \dots & \mu_N w_N \\ \mu_2 w_1 & \mu_2 w_2 & \dots & \mu_N w_N \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1 w_1 & \mu_2 w_2 & \dots & \mu_N w_N \end{pmatrix}, \quad \mathbf{s}_g^+ = \epsilon B_\lambda(T_g) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \quad (20)$$

where  $T_g$  equals the surface or ground temperature. Note that this term will be negligible for shortwave wavelengths.

### 1.5.2 Specular surface optical properties

For a specular surface, the surface emissivity is typically specified as a function of the zenith angle, so  $\epsilon = \epsilon(\mu)$ , and the surface reflectance matrix will be diagonal:

$$\mathbf{R}_g = \begin{pmatrix} 1 - \epsilon(\mu_1) & 0 & \dots & 0 \\ 0 & 1 - \epsilon(\mu_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 - \epsilon(\mu_N) \end{pmatrix}, \quad \mathbf{s}_g^+ = B_\lambda(T_g) \begin{pmatrix} \epsilon(\mu_1) \\ \epsilon(\mu_2) \\ \vdots \\ \epsilon(\mu_N) \end{pmatrix}. \quad (21)$$

## 1.6 Adding in the surface and the upper boundary

Now we have reduced our problem to the reflection & transmission matrices for the full atmosphere, and the upwelling and downwelling sources for the full atmosphere  $\mathbf{s}^+$  and  $\mathbf{s}^-$ . The picture we have is a single (atmospheric) layer, with the surface below (potentially emitting *and* reflecting radiation) and space above (emitting downwelling radiation upon our atmosphere - the cosmic microwave background). Again using the interaction

principle, but this time for a surface reflectance matrix  $\mathbf{R}_g$  and surface upwelling source  $\mathbf{s}_g^+$ , we can find the upwelling radiance vector at TOA  $\mathbf{s}_{TOA}^+$  and the downwelling radiance vector at the surface  $\mathbf{s}_g^-$ .

We can even include the incoming emission from cold space, which is essentially the isotropically downwelling cosmic microwave background radiation. We'll write this as  $\mathbf{s}_{TOA}^-$

$$\mathbf{s}_{TOA}^- = B_\lambda(T_{cmb}) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad (22)$$

where  $T_{cmb} = 2.625K$ . This term only need be included in the microwave; otherwise, it will be completely negligible.

Letting the reflectance matrix for radiation incident from above the layer be denoted  $\mathbf{R}_{ab}$  and from below the layer as  $\mathbf{R}_{ba}$ , and similarly for the transmission matrix, we have

$$\mathbf{s}_{TOA}^+ = \mathbf{s}^+ + \mathbf{R}_{ab}\mathbf{s}_{TOA}^- + \mathbf{T}_{ba}(\mathbf{I} - \mathbf{R}_g\mathbf{R}_{ba})^{-1}(\mathbf{s}_g^+ + \mathbf{R}_g(\mathbf{s}^- + \mathbf{T}_{ab}\mathbf{s}_{TOA}^-)) \quad (23a)$$

$$\mathbf{s}_g^- = (\mathbf{I} - \mathbf{R}_{ba}\mathbf{R}_g)^{-1}(\mathbf{s}^- + \mathbf{T}_{ab}\mathbf{s}_{TOA}^- + \mathbf{R}_{ba}\mathbf{s}_g^+) \quad (23b)$$

## 1.7 Final intensities & fluxes

For the purpose of calculating upwelling intensity at TOA and downwelling intensity at the surface, we're done. Remember we explicitly included a "stream with zero weight" for the desired zenith angle in our  $\boldsymbol{\mu}$  vector, given by  $\mu_u$ . Because we added it as the last element of  $\boldsymbol{\mu}$ , the desired upwelling TOA intensity  $I_{TOA}^\uparrow(\mu_u)$  is given by

$$I_{TOA}^\uparrow(\mu_u) = s_{TOA}^+(N_s + 1) \quad (24)$$

and the desired downwelling intensity at the surface is  $s_g^-(N_s + 1)$ . However, we can also the corresponding upwelling TOA flux  $F_{TOA}^+$  and  $F_g^-$ . Again using our quadrature rule to integrate over  $\mu$ , it is easily shown that

$$F_{TOA}^+ = 2\pi \sum_i s_{TOA}^+(i) \mu_i w_i \quad (25a)$$

$$F_g^- = 2\pi \sum_i s_g^-(i) \mu_i w_i \quad (25b)$$

## 2 Modifications to include a solar source

It may be that we would prefer to work in the shortwave (solar source) rather than in the long wave (thermal) source, or that even we are interested in a wavelength where both terms may be important. Luckily, if you have made it this far, you have done most of the work, and only minor modifications are required to include a solar source.

We will assume that the incoming spectral solar flux is given by  $F_{\odot}$ , which is a function of wavelength  $\lambda$  and has typical units of  $Wm^{-2}\mu m^{-1}$ . Note that this *does not include the*  $\cos\theta_0$  *term!!*

## 2.1 Addition of the outer azimuthal moment loop

The addition of the solar source term breaks the azimuthal symmetry of the plane-parallel radiative transfer problem. If your aim is only to calculate (spectral) fluxes, then what I'm about to say is unnecessary, because up to this point we've explicitly calculated azimuthally-averaged intensities only, and these are all that are required in a flux calculation. As shown in section 6.1.2 of Liou's book, the plane-parallel RT equation can be decomposed into  $M + 1$  independent RT equations, where the total intensity is given by:

$$\mathbf{I}(\tau, \mu, \phi) = \sum_{m=0}^M I_m(\tau, \mu) \cos m(\phi - \phi_0) \quad (26)$$

where  $\phi_0$  is an arbitrary azimuth angle, but we'll take it to equal the solar azimuth angle. It is easily shown that  $I_0(\tau, \mu)$  is the azimuthally-averaged intensity at a given  $\mu$ .

Therefore, what we did in section 2 was to solve only for  $I_0(\mu)$  upwelling at TOA and downwelling upon the surface. We want to expand this to solve for up to azimuthal moment  $m = M = L$  where as before  $L$  is some maximum number of expansion coefficients needed to characterize the scattering phase function. Because  $L$  is related to how forward-scattering the phase function is, the number of times through the loop is essentially related to how large the particles are in the atmosphere. It is therefore fairly easy for  $M$  to become prohibitively large. First it is important to note, however, that if one has very poor angular resolution in the calculation via a relatively small number of streams, then there is no use making  $M$  very large. In general, a good rule of thumb is to set  $M=2 * N_s - 1$  (such that the number of calculated azimuthal moments,  $M + 1$ , equals the number of full streams, which is twice the number of half-streams). For example, in the simplest case of a 2-stream model ( $N_s = 1$ ), one would only calculate for  $m = 0$  and  $m = 1$ .

The second thing to note is that if one really does have large particles (*i.e.* large size parameters) and hence highly forward peaked phase functions, it is very desirable to employ some kind of similarity transformation on the atmospheric optical properties, such as the  $\delta$ -M scaling method of Wiscombe (Wiscombe, 1977b). In that method,  $L = 2N_s$ , *i.e.* the number of full streams.

## 2.2 Modification of layer doubling to include a direct solar source

In general, the direct-beam solar flux density at a vertical level  $\tau = \tau(z)$  in the atmosphere is given by  $F_{\odot}e^{-\tau/\mu_0}$ . This will create a "solar source" term *inside every initial (very thin) doubling layer*, via local scattering. This solar source term will be in addition to the thermal source term. However, because the solar source term decreases as  $\exp -\tau/\mu_0$  throughout the layer, instead of linearly (like the thermal source term), we will treat it completely separately.

### 2.2.1 Solar source thin-layer initialization

The solar source term for the top-most very thin layer (before any doubling occurs) be  $\mathbf{s}_{0,solar}^+$  for upwelling and  $\mathbf{s}_{0,solar}^-$  for downwelling. Again using the Infinitesimal Generator Initialization, these are given by

$$\mathbf{s}_{0,solar}^+(i) = \frac{\tilde{\omega}}{4\pi} \frac{\delta\tau}{\mu_i} p_m^-(i) F_{\odot} e^{-\tau_i/\mu_0} \quad (27a)$$

$$\mathbf{s}_{0,solar}^-(i) = \frac{\tilde{\omega}}{4\pi} \frac{\delta\tau}{\mu_i} p_m^+(i) F_{\odot} e^{-\tau_i/\mu_0} \quad (27b)$$

where  $F_{\odot} e^{-\tau_i/\mu_0}$  is the solar flux incident upon the *top* of layer  $l$ , and we have defined  $\tau_l$  to be the total zenith optical depth from TOA to the top of layer  $l$ . The terms  $\mathbf{p}_m^-(i)$  and  $\mathbf{p}_m^+(i)$  represent the  $m^{th}$  azimuthal moment of the phase function for scattering from solar zenith  $\mu_0$  to emergent direction  $\mu_i$ :

$$p_m^+(i) = \sum_{\ell=m}^L a_{\ell}^m \chi_{\ell} P_{\ell}^m(\mu_i) P_{\ell}^m(\mu_0) \quad (28a)$$

$$p_m^-(i) = \sum_{\ell=m}^L (-1)^{\ell+m} a_{\ell}^m \chi_{\ell} P_{\ell}^m(\mu_i) P_{\ell}^m(\mu_0) . \quad (28b)$$

As usual we require normalization for  $m = 0$ , such that  $\sum_i (p_m^+(i) + p_m^-(i)) w_i = 2$ .

We note that we'll also have to calculate the other initial layer quantities as before.

### 2.2.2 Doubling with thermal and solar sources

For doubling with a thermal and solar source, we begin by computing the thin layer reflectance and transmittance matrices  $\mathbf{R}_0$  and  $\mathbf{T}_0$  appropriate to the particular azimuthal moment  $m$  and atmospheric layer  $l$  we're currently on (equations 9). If  $m = 0$ , from equation 10 we also compute the initial  $\mathbf{y}_0$  and  $\mathbf{z}_0$  vectors, which as stated before correspond to emissivity mean and slope, respectively. We then must also compute the solar source for the top-most initial (thin) layer of this full atmospheric layer, according to equation 28:  $\mathbf{s}_{0,solar}^+$  (upwelling) and  $\mathbf{s}_{0,solar}^-$  (downwelling).

Now that we have the thin-layer initializations of these six quantities, we are ready to undergo our  $N$  doublings. For the four quantities  $\mathbf{R}$ ,  $\mathbf{T}$ ,  $\mathbf{y}$  and  $\mathbf{z}$ , we use equations (14) as before. The value of  $m$  doesn't affect the doubling formulae. The doubling formulae for the upwelling and downwelling solar source are given by:

$$\mathbf{s}_{k+1,solar}^+ = \mathbf{T}_k \mathbf{\Gamma}_k (\mathbf{R}_k \mathbf{s}_{k,solar}^- + \gamma_k \mathbf{s}_{k,solar}^+) + \mathbf{s}_{k,solar}^+ \quad (29a)$$

$$\mathbf{s}_{k+1,solar}^- = \mathbf{T}_k \mathbf{\Gamma}_k (\gamma_k \mathbf{R}_k \mathbf{s}_{k,solar}^+ + \mathbf{s}_{k,solar}^-) + \gamma_k \mathbf{s}_{k,solar}^- \quad (29b)$$

where  $\gamma_k = e^{-2^k \delta\tau/\mu_0}$  is to account for attenuation of the direct beam as we go down through the layer. As before, we perform our  $d$  doublings (such that the initial layer

optical thickness  $\delta\tau$  is rather small, such as 0.01 or less). After doubling, we'll have the total upwelling and downwelling sources from thermal sources (equation 16) and the solar source from equation (29), i.e.  $s_{d,solar}^+$  and  $s_{d,solar}^-$  for each layer in the atmosphere, for a given azimuthal moment  $m$ .

### 2.3 Addition of the solar ground source

After doubling, we can compute the full-atmosphere properties (reflectance and transmission matrices and upwelling and downwelling source vectors) as before, using the recipe given in section 1.4. The formulae do not change, so the same module you made will work again.

Next we must add in the boundary conditions: the surface reflectance matrix and source vector. Because we're still at the same wavelength  $\lambda$ , the surface reflectivity matrix  $\mathbf{R}_g$  remains the same. We now simply must add in the direct-beam solar source to the ground's thermal emission source. That is, a part of the solar beam may make it all the way through the atmosphere and be reflected off the surface. Therefore, in general the surface source has two terms: a thermal source  $\mathbf{s}_{g,thermal}^+$  and a solar source  $\mathbf{s}_{g,solar}^+$ . The thermal term was given in equation (20). Assuming a Lambertian surface albedo of  $\alpha$ , the solar source is given by:

$$\mathbf{s}_{g,solar}^+ = \frac{\alpha}{\pi} F_{\odot} \mu_0 e^{-\tau^*/\mu_0} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad (30)$$

where  $\tau^*$  is the TOA-to-surface zenith optical depth, and again  $\mu_0$  is the cosine of the solar zenith angle.

After this is done, the equations to combine the surface and atmosphere (section 1.6) and calculating final intensities and fluxes (section 1.7) remain essentially the same. The only difference is now we have calculated azimuthal moments of the intensity field for  $m = 0..M$ :  $I_0(\boldsymbol{\mu})$ ,  $I_1(\boldsymbol{\mu})$ , etc. To construct the full total intensity for a given azimuth  $\phi$ , we must use equation (26) given earlier. As before,  $I_0(\boldsymbol{\mu})$  (either upwelling at TOA or downwelling at the surface) represents the azimuthally-averaged intensity, and equations (25) can be used for calculation of the corresponding hemispheric fluxes.

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