Azimuthal Decomposition of the Full RT eqn. ①

Full RT equation with explicit solar source

$$\boxed{\mu \frac{dI(\tau,\vec{\Omega})}{d\tau} = \underbrace{I(\tau,\vec{\Omega})}_{\text{extinction}} - \underbrace{\frac{\tilde{\omega}}{4\pi}\int_{4\pi} I(\tau,\vec{\Omega}')P(\Omega,\vec{\Omega}')\,d\vec{\Omega}'}_{\text{Multiple Scattering Source}} - \underbrace{\frac{\tilde{\omega}}{4\pi} F_{\odot} P(\vec{\Omega},\vec{\Omega}_0)\cdot e^{-\tau/\mu_0}}_{\text{local single-scattering solar source}} + \underbrace{(1-\tilde{\omega})B_\lambda(T(\tau))}_{\text{local thermal source}}} \qquad (1)$$

where $\vec{\Omega} = (\mu,\phi)$ $\quad \Omega_0 = (-|\mu_0|,\phi_0) = (-\mu_0,\phi_0)$

$\vec{\Omega}' = (\mu',\phi')$

- First, recognize this equivalency:

$$\boxed{P(\vec{\Omega},\vec{\Omega}') = \sum_{m=0}^{M}\sum_{\ell=m}^{N} A_\ell^m \chi_\ell P_\ell^m(\mu)P_\ell^m(\mu')\cos m(\phi'-\phi)} \qquad (2)$$

This is the "Addition Theorem for spherical Harmonics"

where $\boxed{A_\ell^m = (2-\delta_{0,m})\frac{(\ell-m)!}{(\ell+m)!}}$

$\delta_{0,m}$: Kronecker-delta Function

$= 0$ for $m \neq 0$

$= 1$ for $m = 0$

- Motivates us to expand intensity field in a similar way (& it turns out to be valid!):

<u>Intensity Azimuthal Expansion</u>

$$\boxed{I(\tau,\mu,\phi) = \sum_{m=0}^{M} I_m(\tau,\mu)\cos(m(\phi-\phi_0))} \qquad (3)$$

- $\boxed{\text{Insert (2) AND (3) into (1)}}$

- So long we'll treat each term separately AND show each one looks like $\sum_m f_m(\mu,\tau)\cdot\cos m(\phi-\phi_0)$

- <u>First Two terms</u> (LHS + extinction)

$$\mu \frac{dI(\tau,\mu,\phi)}{d\tau} = I(\tau,\mu,\phi) - \dots \qquad (4)$$

$$\hookrightarrow \mu \frac{d}{d\tau}\sum_{m=0}^{M} I_m(\tau,\mu)\cos m(\phi-\phi_0) = \sum_{m=0}^{M} I_m(\tau,\mu)\cos m(\phi-\phi_0) - \dots$$

$$\hookrightarrow \boxed{\sum_{m=0}^{M} \mu \frac{dI_m(\tau,\mu)}{d\tau}\cos m(\phi-\phi_0) = \sum_{m=0}^{M} I_m(\tau,\mu)\cos m(\phi-\phi_0) - \dots}$$

- <u>Scattering Term</u>

$$-\frac{\tilde{\omega}}{4\pi}\int_{4\pi} I(\tau,\mu',\phi')\,P(\mu,\phi;\mu',\phi')\,d\mu'\,d\phi' \qquad (5)$$

$$\hookrightarrow = -\frac{\tilde{\omega}}{4\pi}\sum_{m=0}^{M}\sum_{l=m}^{M}\int_0^{2\pi}\int_{-1}^{1} A_l^m \chi_l P_l^m(\mu) P_l^m(\mu') \overset{\cos m(\phi'-\phi)}{\wedge} \sum_{m'=0}^{M} I_{m'}(\tau,\mu')\cos m'(\phi'-\phi_0)\, d\mu' d\phi'$$

$$\hookrightarrow = -\frac{\tilde{\omega}}{4\pi}\sum_{m=0}^{M}\sum_{l=m}^{M}\sum_{m'=0}^{M}\int_{-1}^{1} A_l^m \chi_l P_l^m(\mu) P_l^m(\mu') I_{m'}(\tau,\mu') \underbrace{\int_0^{2\pi}\cos m(\phi'-\phi)\cos m'(\phi'-\phi_0)\,d\phi'}_{\text{easily shown that } =\delta_{m,m'}\,\pi\,(1+\delta_{0,m})\cos m(\phi-\phi_0)}$$

$$\hookrightarrow = -\frac{\tilde{\omega}}{4\cancel{\pi}}\sum_{m=0}^{M}\sum_{l=m}^{M}\cancel{\pi} A_l^m (1+\delta_{0,m})\left[\int_{-1}^{1}\chi_l P_l^m(\mu) P_l^m(\mu') I_m(\tau,\mu')\,d\mu'\right]\cos m(\phi-\phi_0)$$

$$= \sum_{m=0}^{M}\left\{-\frac{\tilde{\omega}}{4}(1+\delta_{0,m})\int_{-1}^{1}\tilde{P}_m(\mu,\mu')\, I_m(\tau,\mu')\,d\mu'\right\}\cos m(\phi-\phi_0)$$

where $\tilde{P}_m(\mu,\mu') \equiv \sum_{l=m}^{M} A_l^m \chi_l P_l^m(\mu) P_l^m(\mu')$

- <u>Local Solar Source</u>

$$-\frac{\tilde{\omega}}{4\pi} F_{\odot} P(\mu,\phi,-\mu_0,\phi_0)\, e^{-\tau/\mu_0} \qquad (6)$$

$\downarrow$ Adding Theorem

$$= -\frac{\tilde{\omega}}{4\pi} \sum_{m=0}^{M} \sum_{\ell=m}^{M} A_\ell^m \chi_\ell P_\ell^m(\mu) P_\ell^m(\mu_0) \cos m(\phi-\phi_0)\, e^{-\tau/\mu_0}$$

$$= \boxed{\sum_{m=0}^{M} \left\{ -\frac{\tilde{\omega}}{4\pi} \tilde{P}_m(\mu,\mu_0)\, e^{-\tau/\mu_0} \right\} \cos m(\phi-\phi_0)}$$

where $\tilde{P}_m(\mu,\mu')$ is the same function as before.

- <u>Local Thermal Source</u>

$$= (1-\tilde{\omega})\, B_\lambda(\tau) \qquad (7)$$

$$= \boxed{\sum_{m=0}^{M} \left\{ (1-\tilde{\omega})\, B_\lambda(\tau)\, \delta_{0,m} \right\} \cos m(\phi-\phi_0)}$$

- So all terms look like

$$\sum_{m=0}^{M} f(\tau,\mu) \quad \cos m(\phi-\phi_0)$$

- Therefore we have $M+1$ independent equations, one for each $I_m(\tau,\mu)$ intensity function.

- Each equation looks like

$$\boxed{\begin{aligned}\mu \frac{dI_m(\tau,\mu)}{d\tau} &= I_m(\tau,\mu) - \frac{\tilde{\omega}}{4}(1+\delta_{0,m})\int_{-1}^{1}\tilde{P}_m(\mu,\mu')\, I_m(\tau,\mu')\,d\mu' \\ &\quad - \frac{\tilde{\omega}}{4\pi}\tilde{P}_m(\mu,\mu_0)\, e^{-\tau/\mu_0} - (1-\tilde{\omega})\,\delta_{0,m}\, B_\lambda(\tau)\end{aligned}} \qquad (8)$$

Full RT equation for azimuthal moment m

where again $\tilde{P}_m \equiv \sum_{l=m}^{M} A_l^m \chi_l P_l^m(\mu) P_l^m(\mu')$ AND

$$I(\tau,\mu,\phi) = \sum_{m=0}^{M} I_m(\tau,\mu) \cos m(\phi-\phi_0)$$

- It is easily shown that

$$\frac{(1+\delta_{0,m})}{2}\tilde{P}_m(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} P(\mu,\phi,\mu',\phi') \cos m(\phi'-\phi)\, d(\phi'-\phi)$$

AND Hence for $m=0$

$$\boxed{\tilde{P}_0(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} P(\mu,\phi;\mu',\phi')\, d(\phi'-\phi) = \sum_{l=0}^{M} \chi_l P_l(\mu) P_l(\mu')} \qquad (9)$$

Azimuthally Averaged value of Phase Function!

- What is the Azimuthally-Averaged Intensity?

$$\tilde{I}(\tau,\mu) = \frac{1}{2\pi}\int_0^{2\pi} \overbrace{\sum_{m=0}^{M} I_m(\tau,\mu)\cos m(\phi-\phi_0)}^{I(\tau,\mu,\phi)}\, d\phi$$

$$= \frac{1}{2\pi}\sum_{m=0}^{M} I_m(\tau,\mu) \underbrace{\int_0^{2\pi}\cos m(\phi-\phi_0)\,d\phi}_{=2\pi\,\delta_{0,m}}$$

$$\boxed{\tilde{I}(\tau,\mu) = I_0(\tau,\mu) = \text{Azi-Avg'd Intensity!}}$$

• So RT equation for $\underbrace{\text{AA'd intensity}}_{I_0(\tau,\mu)}$ is (8) with $m=0$:

$$\boxed{\mu \frac{dI_0(\tau,\mu)}{d\tau} = I_0(\tau,\mu) - \frac{\tilde{\omega}}{2}\int_{-1}^{1} \tilde{P}_0(\mu,\mu') I_0(\tau,\mu')\,d\mu' - \frac{\tilde{\omega}}{4\pi}\tilde{P}_0(\mu,\mu_0)e^{-\tau/\mu_0} - (1-\tilde{\omega})B_\lambda(\tau)} \quad (10)$$

Azimuthally-Avg'd RT eqn

• What do we need to calculate fluxes?

E.g. $F^\uparrow(\tau)$

$$F^\uparrow(\tau) = \int_{\phi=0}^{2\pi}\int_{\mu=0}^{1} I^{\#}(\tau,\mu,\phi)\,\mu\,d\mu\,d\phi$$

$$= \int_{\mu=0}^{1} \underbrace{\int_{\phi=0}^{2\pi} I(\tau,\mu,\phi)\,d\phi}_{=2\pi\tilde{I}(\tau,\mu)}\ \mu\,d\mu$$

$$\boxed{F^\uparrow_{dif}(\tau) = 2\pi\int_0^1 \tilde{I}(\tau,\mu)\,\mu\,d\mu}$$ Upward Flux at vertical location τ (11a)

AND similarly

$$\boxed{F^\downarrow_{dif}(\tau) = 2\pi\int_{-1}^{0} \tilde{I}(\tau,\mu)\,\mu\,d\mu}$$ Downward Flux at vertical location τ (excluding direct solar beam) (11b)

AND for completeness

$$\boxed{F^\downarrow_{direct}(\tau) = F_\odot\,\mu_0\,e^{-\tau/\mu_0}}$$ Downward Flux @ τ (Direct Solar Beam) (11c)

• (Note that one can define the direct solar intensity as

$$\boxed{I^\downarrow_\odot(\tau,\mu,\phi) = F_\odot\,e^{-\tau/\mu_0}\,\delta(\mu-\mu_0)\,\delta(\phi-\phi_0)} \quad (12)$$

where $\delta(x)$ = Dirac-Delta Function such that $\int_{-\infty}^{\infty} f(x)\delta(x-x_0)\,dx = f(x_0)$
)

Quadrature Integration

- In $m^{\underline{th}}$ RT eqn for $I_m(\tau, \mu)$ there is this

$$\int_{-1}^{1} \tilde{P}_m(\mu, \mu') \, I_m(\tau, \mu') \, d\mu'$$

- We can approximate this integral with a "quadrature rule"

$$\int_{-1}^{1} f(\mu') \, d\mu' \approx \sum_{j=1}^{N_s} f(\mu_j) w_j \qquad \text{where} \quad \sum_{j=1}^{N_s} w_j = 1 \qquad (13)$$

- We can choose μ's and w's (quadrature points + weights) such that integration is <u>exact</u> if $f(\mu)$ is a polynomial function of order N_s or less.

  This is called <u>Gaussian Quadrature</u>.

- In RT, Gaussian Quadrature on $(0, 1)$ is preferred for $N_s \geq 2$. For $N_s = 1$, there is no best answer but $\mu_1 = \frac{1}{\sqrt{3}}$ is often recommended. $(54.7°)$
  ($\mu_1 = 0.5$ has nice properties as well)
- We choose such that "up-streams" = − "down-streams"
  i.e. $(\mu_1, \mu_2, \ldots)$ up
  $(-\mu_1, -\mu_2, \ldots)$ Down

- A few Quad Choices:

N_s	μ	w
$N_s = 1$	[redacted] $\frac{1}{2}$	1
$N_s = 2$	0.211	½
	0.789	½
$N_s = 3$	0.113	0.278
	0.5	0.444
	0.887	0.278
$N_s = 4$	0.0694	0.1739
	0.3300	0.3261
	0.6700	0.3261
	[illegible]	[illegible]

Gaussian Quadrature on $(0, 1)$ (for $N_s = 2$ to $N_s = 4$)

Notice that $\sum_i w_i = 1$

$\sum_i \mu_i w_i = \frac{1}{2}$ for $N_s \geq 2$

[redacted]